The Arithmetic Teacher

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New Experiments with Multiplication
CATHERINE STERN

Rate Pairs, Fractions, and Rational Numbers
HENRY VAN ENGEN

Why Not Individualize Arithmetic?
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Enrichment for Understanding
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ROBERT C. TOWNSEND

A Journal of
THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

THE ARITHMETIC TEACHER

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THE ARITHMETIC TEACHER

Volume VII Number 8

December 1960

New Experiments with Multiplication[1]

CATHERINE STERN
New York, New York

SOME YEARS AGO, a child came to me for remedial work in arithmetic. She explained her difficulty, "I can remember to say 'five-three-eight' if the word *Add* is at the top of the page, and I can say 'five-three-two' if the word is *Subtract*. Suddenly I am supposed to say 'five-three- . . . ' I don't know what, because the page says *Multiply*. Now I give up."

This child had refused to take another look at multiplication because of a difficulty which had its origin not within herself but in faulty teaching.

Add	*Subtract*	*Multiply*
5	5	5
3	3	3
—	—	—
8	2	15

Senseless rote memorization of single facts without understanding the underlying arithmetical processes proved an insurmountable problem for this child. Adding, subtracting, and multiplying are different operations and the signs $+$, $-$, $\times$ (*plus*, *minus*, *times*) give us definite commands to carry out particular operations within the structure of our number system. The discovery of these fundamental concepts brought organization and meaning to this girl's total study of arithmetic.

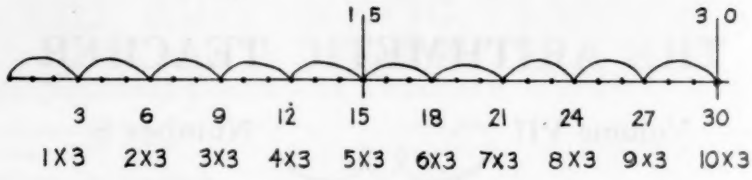
[1] This article grew out of research made possible by a grant of the Carnegie Corporation of New York.

Teachers today all strive to put more meaning into the teaching of arithmetic. In working successfully with this particular child, I found a new approach to multiplication which should also be of interest to the classroom teacher. Even when children do understand multiplication as the operation in which a given number is produced 2 or 5 or 8 times, some members of the class are still troubled when they face learning the 100 multiplication facts. Additional instruction can make these facts of multiplication more easily understood and mastered. I shall explain several techniques that have proved effective in my teaching.

The Use of Scales

One unique way in which children can look at multiplication is based on the idea that the 3's or the 6's or the 7's have scales (Fig. 1, page 382) which are different from the 10-scale but which end at the point at which they intersect the 10-scale.

The 3-scale is shown here as an example of the unforgettable picture which develops as pupils discover the multiples of 3 in our number track. They find that the 3-scale intersects the 10-scale at 30; here the 3-scale reaches its limit. To go beyond 30, we simply add a multiple of 3, as $30+3$, $30+6$, $30+9$ . . . , beginning all over again. The children have only to visualize the peaks between 3 and 30 to find any given multiple of 3. The 10-fact must be included in all



THE 3 - SCALE

FIG. 1

scales; this terminal point is very important for the understanding of the structure of multiplication.

Children need to build the 3-scale and to write discoveries as they are made. Having discovered that ten 3's are 30, the pupils can locate the middle of the scale: five 3's are half of 30, or 15. As they know ten 3's are 30, they know nine 3's will be 30−3, or 27. The other 3-facts are easily related to these fixed points on the scale of 3.

The 6-scale can be taught successfully any time after the 3-scale is mastered, since every second peak on the 3-scale is a peak in the 6-scale (Fig. 2). The peaks of the 9-scale are found at every third peak of the 3-scale.

Similarly, once the 2-scale is mastered, its relation to the 4-scale and 8-scale can be clearly understood and mastered. Since 5 is half of 10, children who understand the 10-scale are ready to master the 5-scale with ease.

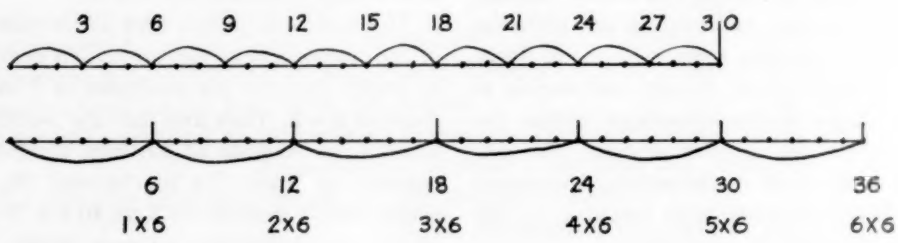
Only the 7-scale remains, with its special characteristics. It is a pleasure to see how rapidly children understand the multiples of 7 by the visual presentation I shall describe later which shows the relationship of the 7's to our decimal system.

Changing Multiples to Tens and Units

In the discussion above, the 9-scale was said to consist of all the third peaks of the 3-scale. This relationship of 9 to 3 is but one way to make mastery of these multiplication facts more easily attained.

A second presentation is to show children how to change each of the multiples of 9 to tens and units. I have constructed the Dual Board with two empty compartments: one for the units and one for the tens. The 10-compartment holds 100 unit-cubes or ten 10-blocks, each 10-block being a fusion of 10 single cubes (Fig. 3, page 383).

Here changing six 9's to 10's and units is illustrated. When children see six 9-blocks in the 10-compartment, they quickly notice that six 9's lack six units to make six 10's. One of the six 9's is taken out of the compartment and placed across the top of the



THE 3- SCALE AND THE 6- SCALE

FIG. 2

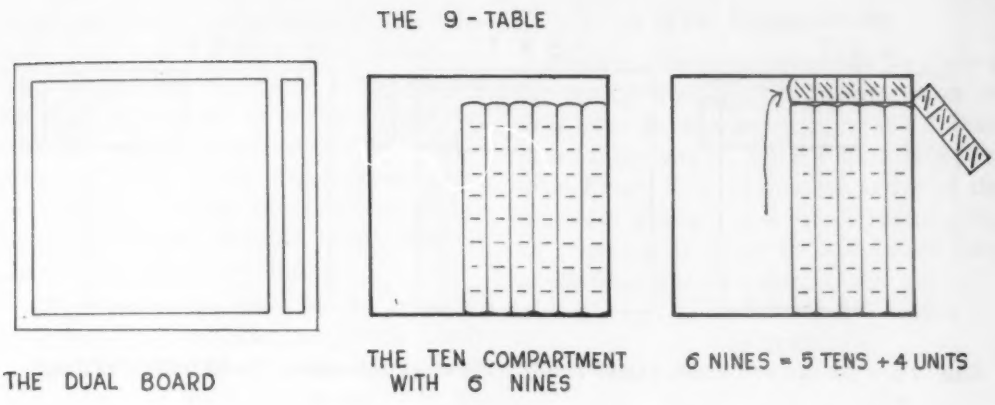


FIG. 3

five 9's left in the compartment so that each of the five 9's assumes the value of a 10 and is converted to our decimal system. The children discover that four units of the horizontal 9-block are in excess of the five 10's; in this way they see that six 9's are five 10's and four units, or 54.

As pupils experiment with the 9's, they gain valuable concomitant understandings. They see why in each multiple of 9 (18, 27, 36, 45, 54, 63, 72, 81, 90) the sum of the digits must be 9. Moreover, exp. imentation with the blocks can clarify the operation of "casting-off 9's."

It is of first importance that every child in the classroom be aware that each number fact of the multiplication table is a conversion of a given number of 9's or 3's or 7's into 10's and units, the standard denominations of our decimal system. The knowledge of this structure enlightens and makes simple the comprehension of multiplication.

The 7-scale was cited above as having special characteristics apparently unrelated to the other number scales. Surprisingly enough, it has one very close relative. A sentence in W. W. Sawyer's thought-provoking book, *Prelude to Mathematics*, is the clue: "The final digits of the 7×table read backwards are of course those of the 3×table."[2]

[2] (Baltimore: Penguin Books, Inc., 1955) p. 20.

This becomes apparent if we compare the multiples of 7 in descending order with the multiples of 3 in ascending order:

Descending 7's:	63	56	49	42	35	28	21	14	7
Ascending 3's:	3	6	9	12	15	18	21	24	27

We see that the amazing statement is true: the digits in both tables correspond. The reason for this relationship only becomes apparent if we think of the multiples of 7 in terms of 10's and units (Fig. 4).

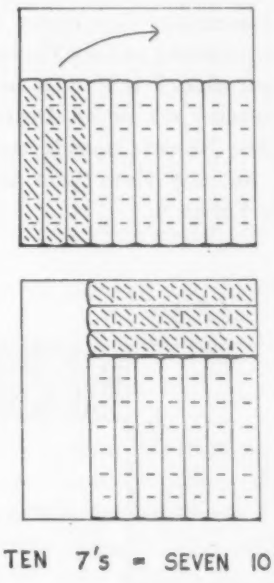


FIG. 4

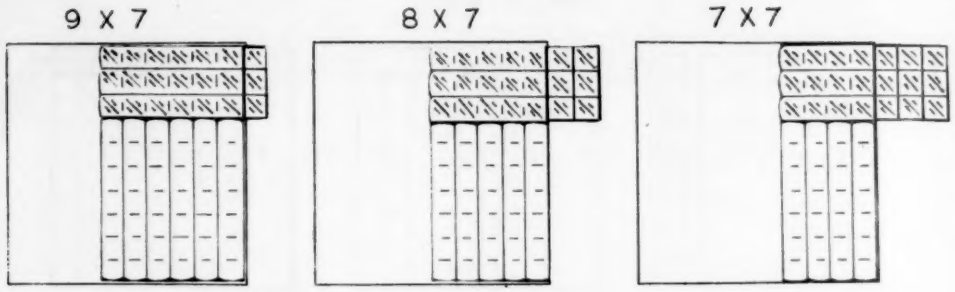


FIG. 5

To change ten 7's into 10's in the Dual Board, we have to move up three 7-blocks, thus proving that ten 7's equal seven 10's, or 70.

If we want but nine of the 7's, we remove a 7-block, as in Figure 5. The result is six 10's and three units, or 63. Removing another 7-block, we see that eight of the 7's are five 10's and (2×3) six units, or 56. Next, seven 7's are four 10's and (3×3) nine units, or 49.

As each additional 7 is removed, an additional 3 is left. If, for example, we want to find what 4 times 7 is, we must push three 7's up horizontally. This leaves one 10 in the 10-compartment and six 3's outside. The result of four times 7 is 10+18, or 28. With 3×7, we reach 7×3, or 21. Instead of seeing the three 7-blocks outside the 10-compartment, we may view them as seven 3-blocks. See Figure 6.

Though we would never teach this procedure beyond 3×7 in the classroom, we shall quickly show how the lower end of the 7-table also relates to the 3-table.

3×7=21	7×3=21
2×7=14	8×3=24
1×7= 7	9×3=27

This correspondence becomes visible if we concentrate on eight 3's and nine 3's and look for the 7's that are concealed in the 3's.

Figure 6 shows that eight 3's minus 10 may be viewed as two 7's (24−10=14). This explains why 8×3 has the same final digit as 2×7. Finally, nine 3's hide one 7-block. We see that nine 3's (or 27) minus 20 equal 7.

The children know that two 7's are 14 and that eight 3's are 24, and it is clear to

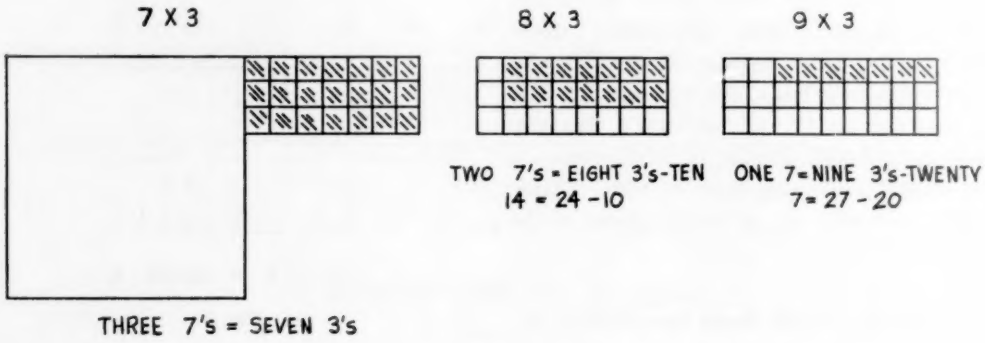


FIG. 6

them that one 7 corresponds to the final digit in 27, which is 9×3. They would not need the explanation offered to the teacher in Figure 6; after all, we as teachers are not satisfied with knowing "facts" in mathematics. We like to see how an experiment undertaken to show the relationship between the 7-table and the 3-table may be carried out to the very end.

When the 7-table was demonstrated with the Dual Board to a class, the children were fascinated. After only a few facts were shown, one of them volunteered, "If you write the 7-table backwards, you get the same endings as in the 3-table frontways."

In the experiment just described, the multiples of 7 were revealed as constructed of a number of 10's increased each time by multiples of 3, as e.g., $60+(1\times3)$, $50+(2\times3)$, $40+(3\times3)$, etc. As 7 needs 3 to make 10, so 9 needs 1. Another look at the multiples of 9 in Figure 3 shows us the same principle: the multiples of 9 are constructed of 10's plus multiples of 1. We find: $80+(1\times1)$, $70+(2\times1)$, $60+(3\times1)$, $50+(4\times1)$, and so on. We now expect the same structural principle to link the multiples of 8 with those of 2, the multiples of 6 with those of 4, and so forth. That this is true becomes evident if we compare the multiples of one table in descending order with the multiples of its complement in ascending order.

Descending 8's:	72	64	56	48	40	32	24	16	8
Ascending 2's:	2	4	6	8	10	12	14	16	18
Descending 6's:	54	48	42	36	30	24	18	12	6
Ascending 4's:	4	8	12	16	20	24	28	32	36

Finally, since $5+5$ is 10, the 5-table will be its own complement, and the ascending and descending order of multiples should show the same units.

Descending 5's:	45	40	35	30	25	20	15	10	5
Ascending 5's:	5	10	15	20	25	30	35	40	45

A New Experiment

Another interesting procedure for finding the multiples of a given number takes its start from the 10-compartment of the Dual Board filled with ten horizontal rows of 10's, as in Figure 7. This time we arrive at the multiples of 9 or 8 or 6 by subtraction, i.e., by removing 10's. Let us study a few facts of the 8-table in this way.

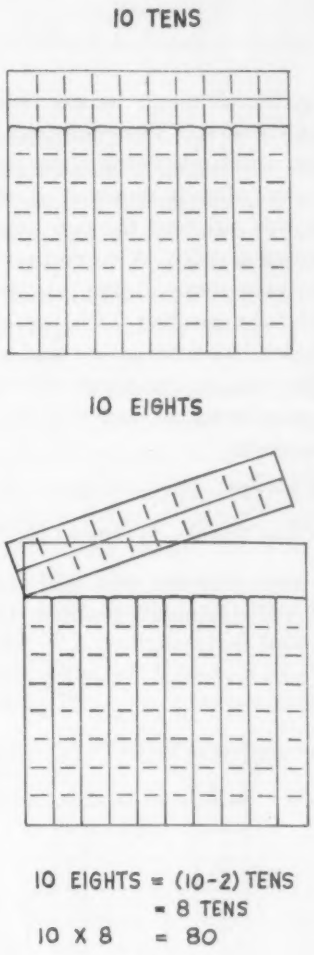


FIG. 7

If we lift out two 10's from the ten 10's in the compartment, we have eight 10's left, which we can replace by ten 8's. We see: $10\times8=$ eight 10's, or 80. If we want only nine 8's, we must remove one 8. Since the new experiment is based on "subtracting 10's," we boldly subtract *one* 10 instead

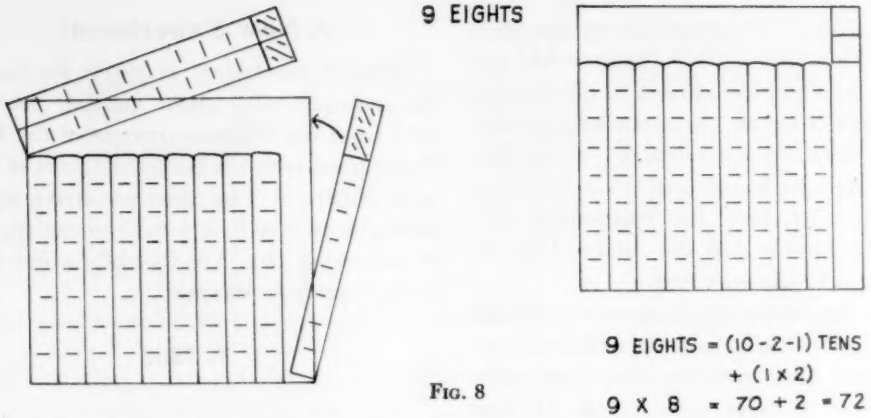


FIG. 8

(see Fig. 8), although we are doing something "forbidden." Two units, or the 1×2 rectangle which was in the upper right-hand corner, were already removed when we took the two 10's out, and the two units cannot be subtracted twice. A correction must be made by once more adding this piece.

To find the product 9×8, we have subtracted two 10's from above and one 10 at the right; that leaves seven 10's to which (1×2) must be added, which results in 72.

Thus, where

$$10\times8=(10-2)\text{ tens}=80$$

$$9\times8=(10-2-1)\text{ tens}+(1\times2)=72.$$

This is actually the structural representation of the binomial product $(10-1)\times(10-2)$, which the pupils will study at a later grade.

Let us go one step further to 8×8, as seen in Figure 9. We have removed the two 10's on top to reach 10×8. Now, removing two more 10's to get an approximation of 8×8, we know that we must correct this act by adding the little square 2×2 which has now been subtracted twice.

$$8\times8=(10-2-2)\text{ tens}+(2\times2)$$

$$=60+4=64$$

This is the structural representation of the square of the binomial $(10-2)$ and illustrates precisely the algebraic expression.

$$(a-b)^2=a^2-2ab+b^2$$

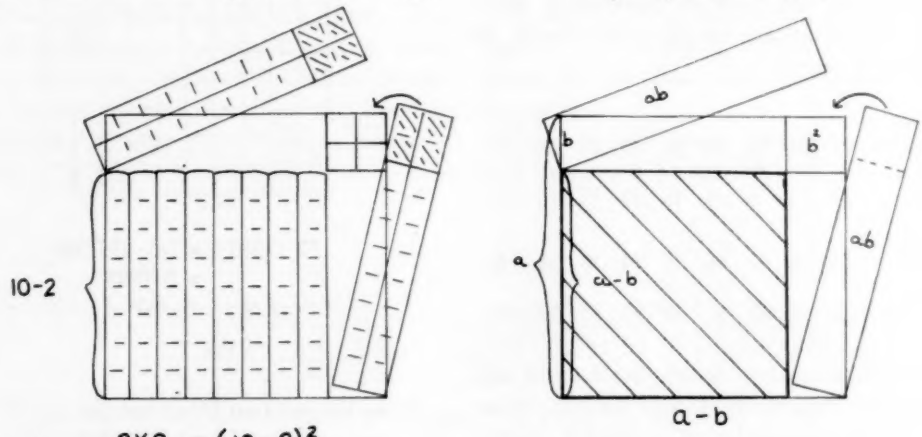


FIG. 9

This procedure can be extended to complete the 8-table and is equally applicable to all multiplication tables. If we want to find the answer to 8×6, we know that we must subtract four 10's (to change 10's to 6's) and two more 10's since we want *eight* of the 6's. The product will be $(10-4-2)$ 10's$+(4\times2)$, or 48. With the addition of 4×2, we put back the little rectangle (the "forbidden rectangle") and thus corrected the mistake that was caused by our insistence on "subtracting 10's."

Explanation of an Old Multiplication Procedure

The operation described above throws light on an old multiplication procedure which works to each pupil's delight, but which can be explained only after the above experiments are understood. This is Dantzig's version of the story:

> To this day the peasant of central France (Auvergne) uses a curious method for multiplying numbers above 5. If he wishes to multiply 9×8, he bends down 4 fingers on his left hand (4 being the excess of 9 over 5), and 3 fingers on his right hand $(8-5=3)$. Then the number of the bent-down fingers gives him the tens of the result $(4+3=7)$, while the product of the unbent fingers gives him the units $(1\times2=2)$.[3]

But why should the "excess over 5" be used as a clue for getting correct results? A closer look at the finger-structure of our two hands will make clear the relationship of the peasant's method to the decimal system. While he uses the bent-down fingers in his computation and determines the number to bend down by the excess of his multiplicand over 5, the *important number is not the number of fingers bent down, but the number of fingers kept up.*

The peasant starts with ten bent-down fingers which represent ten 10's (as in Fig. 10). We did the same when we started our computation with putting ten 10's into the Dual Board in Figure 7. The peasant follows the same method which we demonstrated above with the 8-table in the Dual Board, he subtracts 10's from ten 10's.

[3] Tobias Dantzig, *Number, The Language of Science* (New York: Macmillan, 1941), p. 11.

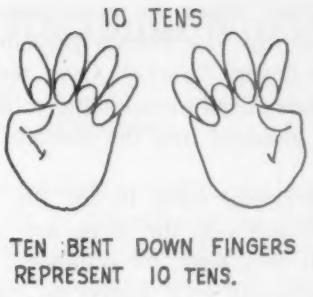


FIG. 10

On hearing "9×8," up go 1 finger on the left hand and 2 fingers on the right hand (9 equals 10 minus 1, and 8 equals $10-2$). One look at the bent-down fingers (shown in Figure 11) gave him the number of 10's: 70. In subtracting 10's on both sides, the peasant made exactly the same mistake as we did (see Figure 8). To correct it, we put back the "forbidden rectangle." The peasant parallels our action by multiplying the numbers represented by the raised fingers: he, too, adds (1×2) to the 70 which the bent-down fingers indicate, thus arriving at the correct result: 72. This means that, from the viewpoint of the mathematician, he is transposing—with a flash of his fingers—the multiplication fact "9×8" into the product of $(10-1)$ $(10-2)$; the answer shows the steps of his technique.

$$(10-1)(10-2)=(10-1-2)\text{ tens}+(1\times2)$$

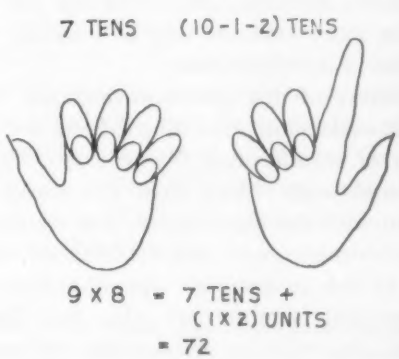


FIG. 11

This then shows the mathematical explanation why the French peasant's play with his fingers "works." Over the years, the mathematically meaningful subtraction from 10 changed into the obscure "excess over 5."

If the pupils want to try the French peasant's method, the steps are easy to follow. If they hear, for instance, "8×6," up immediately go 2 fingers on one hand (since 8 is 10−2) and 4 fingers on the other (since 6 is 10−4). Their hands then look like those in Figure 12.

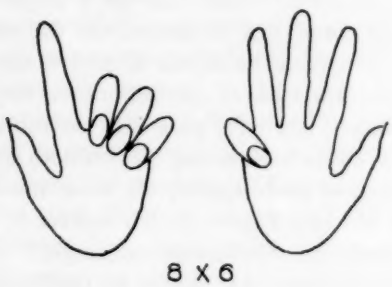


FIG. 12

At a glance, the bent-down fingers show four 10's (the remainder of 10−2−4); the raised fingers lead to the multiplication (2×4), whose product is 8. Almost instantly the result appears as 48.

Our elementary school children do not need the French peasant's method of multiplication because they know a much swifter method that does not require the use of fingers. The method is an interesting experiment, however, and serves the teacher as one more concrete way of showing the process of multiplication.

Mastery of multiplication demands that pupils understand that all products are the result of transforming 8's or 3's or 7's to 10's and units. More than this could be shown with the Dual Board. For example, in experiments with the 8-table, we were able to link an algebraic operation with its geometrical counterpart. In the Dual Board, the children see that the rectangle formed by seven 8's contains 56 units, so that, later, they will readily understand why the area of a rectangle 7 feet by 8 feet is found by multiplication. Before entering the abstract stage of thinking, the child must experiment with concrete materials that show structural relations. The ability to visualize mathematical relations is not a gift of the "born mathematician" alone; each child can and must develop such insight from kindergarten on.

Our function as teachers is to show the inherent perfection of mathematical thinking that we may lead children beyond mechanical memorization of "tables" to the point where they perceive the structural relationships underlying the written symbols.

EDITOR'S NOTE: One of the practicing teacher's most frequent questions is that of teaching the multiplication tables. This article, by a well-known educational theorist and author, should be invaluable as a reference, both as to specific treatment of multiplication and as an example of the application of discovery to the teaching and learning of arithmetic.

Help on Homework

When parents help with math homework, the ends don't justify the means. That was the lesson learned by a West Virginia PTA that recently completed a special, self-financed course in how to work arithmetic problems the modern way. Students at Monroe Elementary School, Huntington, West Virginia, usually managed to get the right answers to their homework problems, but—if their parents helped—they got them the wrong way. Their PTA called on Dean D. Banks Wilburn of Marshall College to tutor the older generation on modern methods of arithmetic teaching. On "graduating" from the nine-hour course, a parent noted, "It wasn't that we were right or wrong, but how we worked the problem."

Taken from Education Summary, *October* 27, 1960, *page* 2.

Rate Pairs, Fractions, and Rational Numbers

HENRY VAN ENGEN
University of Wisconsin, Madison, Wisconsin

Introduction

MOST ELEMENTARY-SCHOOL TEACHERS are aware that intensive efforts are being made to modernize the arithmetic program. In addition to introducing new mathematical ideas, there are suggestions for changing the terminology of arithmetic to make it conform to present-day usage. Even the familiar "whole" numbers of arithmetic have been included in the list of things to be renamed. Fractions, particularly, have received the close scrutiny of the "rewrite" groups. Some groups define a fraction as a symbol for a rational number. For other groups, a fraction is the quotient of two "whole" numbers. Frequently the two terms, "fraction" and "rational number," are used interchangeably.[1]

Except for the insecure arithmetic teacher, the problem of naming the various sets of numbers found in the elementary school curriculum is not a serious one. However, the confusion of names and ideas appearing in the literature, and in some instances within one document, may be an indication of a more serious problem. One of the important characteristics of a good curriculum is that of continuity—a smooth flow of ideas. This is also an important characteristic of good mathematics. To find this smooth flow of ideas, one must look to mathematical structures to see how a mathematician uses these ideas. One will find that there are several acceptable systems of ideas, each of which could serve as a guide for building a sequence for the elementary school.

Choosing from among the various acceptable ways to characterize the numbers of arithmetic[2] is not the only problem faced by curriculum constructors. It is trite to say that a real teacher must always keep in mind that the children and their thought processes possess characteristics. Some of these characteristics are well known. The fundamental problem here, then, is to select an approach for systematizing numbers that is in harmony with mathematics as well as in harmony with the thought processes of children. The mathematics of the schools is frequently not in harmony with contemporary mathematics. It would be too bad to recommend that the schools of today adopt a curriculum in harmony with mathematics but not in harmony with the essential thought processes of children. We need both harmonies, of course. However, the choices are not as simply made as some persons seem to think. It will be the purpose of this paper to develop from the various mathematical systems a program of studies for introducing the use of ordered pairs of numbers in the elementary and junior high schools. This choice must be mathematically sound and psychologically defensible.

For purposes of this paper, we will call the set of numbers 0, 1, 2, 3, 4, 5, 6, · · · the natural numbers. Usually, these are called the "whole" numbers by elementary school teachers. We will assume that the child is familiar with the natural numbers, some of their properties, and the properties of the operations on natural numbers. On this basis, we will investigate some uses of ordered pairs of natural numbers and their place in the curriculum.

[1] The School Mathematics Study Group uses the "symbol for a number" definition of a fraction. In the next breath, they speak of "multiplying numerator and denominator of a fraction by two." This raises the question as to whether multiplication is an operation on symbols, symbols and numbers, or numbers. SMSG must multiply a symbol and a number if it persists in using the "symbol for a number" definition and the standard terminology.

[2] For the somewhat more sophisticated reader, the phrase "the numbers of arithmetic" can be interpreted as meaning the nonnegative rational numbers of mathematics.

Rate Pairs

In the early grades, the child has acquired many of his mathematical ideas from situations that require the use of two numbers and an operation. For example, he has learned that the symbol 5+2 is associated with all instances in which a set of two things has joined a set of five things. From situations of this type, the child abstracts the idea of addition as a mathematical operation. After having acquired a concept of the four fundamental operations, the child meets situations that are described through the use of two natural numbers without involving a mathematical operation. Consider the following situations:

a. Jim and his younger brother, John, earn money by mowing lawns. They decide that it is only fair to divide their earnings at the rate of \$3 for Jim to every \$2 for John.

b. Bill bought three neckties for \$2.

c. Jane and Sue packed apples for Christmas baskets. They placed six apples in each basket.

These and a multitude of similar situations do not imply the use of any mathematical operation previously experienced by the children. The situations describe what we call a ***rate***. A rate is a physical situation in which we make a many-to-many correspondence. Thus, three ties for two dollars suggests a 3-to-2 correspondence. Obviously, there are situations that suggest a many-to-one, or a one-to-many, correspondence. Thus, six apples in each box suggests a 6-to-1 correspondence, while one box for six apples is usually expressed as a 1-to-6 correspondence. Here the order of the groups of words "six apples" and "each box," and "one box" and "six apples," suggests the 6-to-1 or the 1-to-6 correspondence. Later we will see that while either 6-to-1 or 1-to-6 may be associated with a "six apples in each box" situation, there is an important difference between the mathematics of the 1-to-6 and the 6-to-1 ideas.

While it would be possible to use such symbols as 5-to-6 to represent a many-to-many correspondence, it is desirable, from a mathematical point of view, to have a more concise way to represent such rates. We will follow the established convention and use the symbol 5/6 to express a 5-to-6 correspondence. Note that, for the present, the symbol 5/6 is read "5 to 6" or "5 for 6" and *not* five-sixths. We will call pairs of natural numbers, written in the above manner, representing a many-to-many correspondence a ***rate pair***.

The fact that, in the present context, we do not read the symbol 5/6 as five-sixths may cause the reader some concern. Our reason for not reading it as five-sixths is that, in general, number pairs representing rates are not added as we usually think of adding fractions. For example, I buy ribbon at the rate (price) of five yards for \$2. Later I need more of the same ribbon. I find that the price is now three yards for \$1. What price did I pay for the eight yards of ribbon? Obviously, eight yards for \$3. In this situation, we are working with the following rate pairs: 5/2 and 3/1. If we add these rate pairs according to the usual fraction rules, we get nonsense. Thus, $5/2+3/1 = 5/2+6/2 = 11/2$. We most certainly did not buy eleven yards for \$2 or pay \$5.50 per yard.

Because adding rate pairs in the manner of fractions does not help us interpret what goes on in the physical world, we will not call a rate pair a fraction and we will not read a name for a rate pair as a fraction numeral. Rate pairs have important uses and important mathematical properties of their own, but these uses and properties do not coincide entirely with those of fractions. To show you that this is really the case, we will further investigate the properties and uses of rate pairs. Then we will introduce the properties and uses of fractions and rational numbers and make some important contrasts between rate pairs and fractions. All of this will have an important bearing on the work of the elementary school and the junior high school.

Properties of Rate Pairs

To introduce the study of rate pairs in the fourth grade, we might use the following

sequence of ideas. If Mary and Sue divide their earnings at the rate of \$2 to \$1, what would be each girl's share if they earned \$24 for the summer? The first method that yields a solution might well involve taking 24 objects and sharing them at the rate of two objects to one object. The result: Mary gets \$16 and Sue gets \$8. What would be the result if 4-to-2 represents the rate at which Mary and Sue shared their earnings? Again we find that Mary gets \$16 and Sue gets \$8. What happens if Mary and Sue share them at the rate of \$8-to-\$4?

Next, consider a situation in which the price (rate) of buttons is three buttons for 5¢. Manipulation of buttons and play money will show that three buttons for 5¢ is the same price as six buttons for 10¢. Also, it is the same as nine buttons for 15¢; twelve buttons for 20¢; and so on.

From these physical situations, the child learns that rate pairs occur in sets. For the price-of-buttons situation we can write the following:

$\{3/5, 6/10, 9/15, 12/20, 15/25, 18/30, \cdots\}$

The three dots mean that there is no end to the rate pairs that belong to the set. It is important to note that each rate pair used in the price-of-buttons situation is interpreted to mean "*the price of the buttons.*" The same set of rate pairs could be used for other purposes, for example, to study Great Britain's wheat import situation—that is, Great Britain imports three bushels of wheat for each five bushels of wheat consumed.

Since all these rate pairs represent the same price, or rate, we will say that the members of the above set are equivalent.[3] This agreement enables us to write the following endless set of sentences.

$3/5 \sim 6/10;\quad 6/10 \sim 9/15;\quad 3/5 \sim 9/15;$

$6/10 \sim 12/20;\quad 3/5 \sim 30/50; \cdots$

We read $3/5 \sim 6/10$ as "3-to-5 is equivalent to 6-to-10."

Through our study of rate pairs, we note that sets of rate pairs have the following properties. We can use the set tabulated below to convince ourselves that the properties hold in a particular instance.

$\{1/2, 2/4, 3/6, 4/8, 5/10, 6/12, 7/14, \cdots\}$

a. If two rate pairs belong to the same set, that is, if they are equivalent, then their cross-products are equal.

Symbolically: If $a/b \sim c/d$, then $ad = bc$.

Example: $3/6 \sim 42/84$ because $3 \times 84 = 6 \times 42$

b. If in a given set of rate pairs we multiply each component of a rate pair[4] by a natural number, then we get another member of the same set of rate pairs.

Symbolically: $\frac{a}{b} \sim \frac{na}{nb}$ Example: $\frac{2}{3} \sim \frac{5 \times 2}{5 \times 3} = \frac{10}{15}$

c. If we divide both terms of a rate pair by a common factor, we obtain another member of the set of rate pairs.

d. Any given rate pair (except 0/0) belongs to one and only one set of rate pairs. (From the physical situation point of view, this is a comforting thought. If a rate pair associated with a price situation belonged to more than one set of equal rate pairs, things would be very confused.)

Let us see how these properties make rate pairs very useful for problem solving.

Using Rate Pairs

Having introduced the above properties of rate pairs, we can now work many problems more easily than we could with the usual techniques introduced in the elementary school. Here are a few examples.

a. If four apples cost 15¢, what is the cost of twelve apples?

$4/15 \sim 12/N$

$4 \times N = 15 \times 12$ (Cross-product principle)

$N = 45$ (Inverse-operations principle)

b. Mr. Jones traveled 450 miles in nine hours. What was his average rate per hour?

$450/9 \sim N/1$

$450 \times 1 = 9 \times N$

$50 = N$

[3] Here we say "equivalent" rather than "equal," as commonly practiced in elementary schools. After allowing for a degree of imprecision, teachers may wish to replace "equivalent" by "equals" if this makes the paper easier for them to read.

[4] By the component of a rate pair we mean the following: Each rate pair is composed of two natural numbers. Each number is called a component of the rate pair, in fact, the first number is the first component of the rate pair and the second number is the second component of the rate pair.

c. Three out of twenty children were sick in a recent flu epidemic in City A. What per cent of the children in City A were sick?

$$3/20 = N/100$$
$$20 \times N = 3 \times 100$$
$$N = 15$$

d. How many quarts in 420 pints?

$$2/1 \sim 420/N$$
$$2 \times N = 420 \times 1$$
$$N = 210$$

e. There are five apples in each of seven boxes. How many apples in all?

$$5/1 \sim N/7$$
$$N = 5 \times 7$$

Interpreting Sets of Rate Pairs

As a teacher you should notice at least one more thing about sets of rate pairs. The two sets of rate pairs shown below can be associated with the same situation, for example, one apple for 2¢ or 2¢ for one apple. They are not, however, the same set of rate pairs.

$$\{1/2, 2/4, 3/6, 4/8 \cdots\}$$
$$\{2/1, 4/2, 6/3, 8/4 \cdots\}$$

Obviously, a price of 2¢ per apple and one apple for 2¢ are the same price even though expressed differently. Each of the sets given above could be used to solve any two-to-one correspondence problem or any one-to-two correspondence problem, but the sets are not the same sets. If they were the same, then each member of one set would be equivalent to any member of the other set. However, the cross-product test readily shows us that 1/2 is not equivalent to 2/1.

Next, we recall for you the properties of fractions as studied in the elementary school.

Fractions

It has already been pointed out that we do not add rate pairs as we add the fractions of ordinary arithmetic. Because of this, we will refuse to call a rate pair a number. However, you will see that we do call fractions numbers. The reason for this will be the object of our next investigation into the interesting ways we use ordered pairs of numbers in mathematics.

Consider the familiar pie so frequently used in teaching arithmetic. Cut this pie into four parts in such a way that all of the parts are of the same size, that is, each part has the same amount of pie in it. At a certain dinner party, three of these parts were eaten. Let us consider the elemental ideas involved in this familiar fraction situation.

In the first place, there is the idea of "cutting the pie into four parts each having the same amount of pie." In the second place, there is the idea of "taking three of the four" parts. The latter is a many-to-many correspondence in its simplest form. The former is an initial step which must be taken to enable us to set up a many-to-many correspondence. (Three-to-four in the illustration above.) We synthesize these two ideas and symbolize the combination by using the symbol $\frac{3}{4}$ and read it *three-fourths*.

In such pie-cutting situations we notice a number of things. We notice that $\frac{2}{4}$ (two-fourths) of the pie is the same amount of pie as $\frac{1}{2}$ (one-half) of the pie. If we continue our investigation, we find that fractions, like rate pairs, belong to equivalence sets. For example, the set shown below is a set of fractions. Each fraction can be associated with situations in which we cut things into four parts of the same amount and then consider two of the parts.

$$\left\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}, \cdots\right\}$$

Just as with rate pairs, we learn

that two fractions $\frac{a}{b}$ and $\frac{c}{d}$ are equivalent

if and only if $ad = bc$. We also learn that fractions belong to equivalence sets and that a given fraction does not belong to more than one equivalence set. In fact, up to this point in our discussion, fractions and rate pairs have the same properties, that is, they obey the same laws.

TABLE I

PROPERTIES OF FRACTIONS*

Addition	Multiplication
1. Adding two fractions always produces another fraction.	1. Multiplying two fractions always produces another fraction.
2.† $\frac{a}{b}+\frac{c}{d}\sim\frac{ad+bc}{bd}$	2. $\frac{a}{b}\times\frac{c}{d}\sim\frac{ad}{bd}$
3. $\frac{a}{b}+\frac{c}{d}\sim\frac{c}{d}+\frac{a}{b}$	3. $\frac{a}{b}\times\frac{c}{d}=\frac{c}{d}\times\frac{a}{b}$
4. $\frac{a}{b}+\left(\frac{e}{d}+\frac{e}{f}\right)\sim\left(\frac{a}{b}+\frac{e}{d}\right)+\frac{e}{f}$	4. $\frac{a}{b}\times\left(\frac{c}{d}\times\frac{e}{f}\right)\sim\left(\frac{a}{b}\times\frac{c}{d}\right)\times\frac{e}{f}$
5. $\frac{a}{b}+0\sim 0+\frac{a}{b}\sim\frac{a}{b}$	5. $\frac{a}{b}\times 1\sim 1\times\frac{a}{b}\sim\frac{a}{b}$
	6. $\frac{a}{b}\times\frac{b}{a}\sim 1$
$\frac{a}{b}\left(\frac{c}{d}+\frac{e}{g}\right)\sim\left(\frac{a}{b}\times\frac{c}{d}\right)+\left(\frac{a}{b}\times\frac{e}{g}\right)$	

* The symbol 3/0 does not represent a fraction. It is an improper symbol since we cannot have fractions of the type $\frac{N}{0}$.

† Here we use the symbol for "equivalent" ($\sim$) since, for example, $\frac{1}{2}+\frac{1}{3}$ and $\frac{5}{6}$ are not two different symbols for the same number pair. In a sense they are different symbols for "equivalent pieces of pie" but not the "same piece of pie."

Now it becomes imperative to show that we can do some things with fractions that we do not do with rate pairs. Before making this contrast, it is desirable to review the properties of fractions.

Properties of Fractions

Just as it makes sense to have the union of two sets, it makes sense to have the "union" of two parts of pies. The problem becomes one of finding out how to use the ordered pairs of numbers, interpreted as fractions, to represent the "union" of parts of pies.

The story of how to use pairs of numbers, called fractions, is all too familiar to anyone who in the past had an elementary education in the schools of America. We will only recall here the fundamental properties of number pairs used as fractions in Table I.

Most certainly, these ideas about fractions and operations with fractions must be taught in the elementary school. At present, however, some of these ideas are not taught in many of the schools of America.

For example, all elementary children should know that for every fraction, such as $\frac{2}{3}$, there is another fraction ($\frac{3}{2}$) such that their product is 1. ($\frac{2}{3}\times\frac{3}{2}\sim 1$) This is a fundamental property of fractions. To fix the ideas in Table I, we will contrast the properties of fractions with the properties of rate pairs.

Fractions and Rate Pairs

It is now apparent why fractions and rate pairs (usually called ratios) are often confused. Both have in common the following properties:

a. The test for equivalence;

b. Membership in only one equivalence set.

Here the similarity ends. Pairs of numbers used as fractions can be added according to the usual rules of arithmetic, but pairs of numbers used as a rate pair are not added according to the *usual* arithmetic rules. However, rules can be devised to add and multiply pairs of numbers as found in rate situations.

There is one important distinction between rate pairs and fractions which may pass unnoticed even to those who are fairly familiar with mathematics.

When working with fractions, we almost always have two or more equivalence sets in mind. For example, the fraction $\frac{1}{2}$ is a member of the first equivalence set shown below and $\frac{2}{6}$ is a member of the second equivalence set.

$$\left\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}, \ldots\right\}$$

$$\left\{\frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}, \frac{5}{15}, \ldots\right\}$$

When we add $\frac{1}{2}$ and $\frac{2}{6}$, we have selected a member from each equivalence set. The process of addition, then, enables us to find a member of a third equivalence set ($\frac{5}{6}$).

$$\left\{\frac{5}{6}, \frac{10}{12}, \frac{15}{18}, \ldots\right\}$$

As a bonus for our efforts, we find that it makes no difference which member of each of the first two equivalence sets are added, the sum will always appear in the third equivalence set.

As we work with rate pairs, we notice that at no time are we working with more than one equivalence set. The rate-pair problems always tell us that we are to work with a given equivalence set (two ties for \$3) and that we are to find another member of this same equivalence set (how many ties for \$15).

This, then, is the key distinction. Fractions always involve two or more equivalence sets. Rate pairs always involve only one equivalence set.

In summary, we can say what we mean by rate pairs and fractions insofar as these terms are used in school mathematics.

Rate Pairs: Consider the set of all possible pairs of natural numbers. This set of pairs is partitioned into equivalence sets by means of the following:

a. If two pairs a/b and c/d belong to the same set, then $ad=bc$, and conversely,

b. If $ad=bc$, then the two pairs a/b and c/d belong to the same set.

Members of these equivalence sets are called *rate pairs.*

Fractions: The set of all pairs of naturals are partitioned into equivalence sets as indicated in the discussion for rate pairs with one exception. We do not allow pairs of the type $\frac{N}{0}$ to occur. But we go further than forming equivalence sets. We require that the pairs of naturals also obey the following conditions.

c. If $\frac{a}{b}$ is a member of one equivalence set and $\frac{c}{d}$ is a member of another equivalence set, not necessarily different from the one $\frac{a}{b}$ belongs to, then a third equivalence set is determined by the pair $\frac{ad+bc}{bd}$, and we call this pair the sum of $\frac{a}{b}$ and $\frac{c}{d}$. Symbolically, we write the following:

$$\frac{a}{b}+\frac{c}{d}\sim\frac{ad+bc}{bd}.$$

d. If $\frac{a}{b}$ is a member of one equivalence set and $\frac{c}{d}$ is a member of another equivalence set, not necessarily different from the one $\frac{a}{b}$ belongs to, then a third equivalence set is determined by the pair $\frac{ac}{bd}$, and we call this pair the product of $\frac{a}{b}$ and $\frac{c}{d}$.

Symbolically, we write the following:

$$\frac{a}{b}\times\frac{d}{c}\sim\frac{ac}{bd}.$$

When pairs of natural numbers obey conditions (a), (b), (c), and (d), we call them *fractions*.[5]

The above properties of pairs of natural numbers can be generalized to pairs of numbers other than natural numbers. We will not, however, make the generalization in this article.

The acceptance of the general program outlined above leads us to reject some widely held points of view regarding the work with fractions in the elementary school. It will be profitable to dwell on just a few of these rejected points of view.

Points of View to Be Rejected

It is quite common to develop the concept of a unit fraction through various visual devices. Following these experiences, children are told that $\frac{1}{4}+\frac{1}{4}+\frac{1}{4}=\frac{3}{4}$. In other words, a fraction such as $\frac{3}{4}$ is established by adding three unit fractions. Or some argue that since $\frac{1}{4}+\frac{1}{4}+\frac{1}{4}=3\times\frac{1}{4}$, then $3\times\frac{1}{4}=\frac{3}{4}$. This approach is full of mathematical inconsistencies and psychological errors. Let us examine just a few.

Remember that we have told children that we are going to study those situations described by pairs of natural numbers and that these pairs are not associated by means of a mathematical operation. Hence, if we are "true to our word," we cannot tell children that $\frac{1}{4}+\frac{1}{4}+\frac{1}{4}=3\times\frac{1}{4}=\frac{3}{4}$. Furthermore, from a mathematical point of view, we have committed a gross error if we use this approach. The children have not been taught how to add or multiply fractions at this point, and yet we are using addition and multiplication to "define" a fraction before the operations on such fractions have been established.

A second mathematical argument for abandoning the usual approach to this topic involves a rather subtle but important mathematical point.

To argue that $\frac{1}{4}+\frac{1}{4}+\frac{1}{4}=3\times\frac{1}{4}$ is bad because we are mixing operations with number pairs and "single" numbers. Actually we should not be allowed to consider $3\times\frac{1}{4}$. We can legitimately ask questions only about the result of multiplying such pairs as $\frac{3}{1}$ and $\frac{1}{4}$, that is, we ask what pair should we associate with the product shown below?

$$\frac{3}{1}\times\frac{3}{4}$$

This is a question entirely in the realm of pairs of numbers and we should phrase all of our questions in these terms because number pairs are the realm of mathematics under consideration. After having studied the operations on pairs of numbers, then children should be led to see that there are certain interesting parallels between operations on certain fractions and operations on natural numbers. Some illustrations of these parallels are exhibited below.

Operations on Fractions		*Operations on Naturals*
$\frac{3}{1}+\frac{4}{1}\sim\frac{7}{1}$	corresponds to	$3+4=7$
$\frac{3}{1}\times\frac{4}{1}\sim\frac{12}{1}$	corresponds to	$3\times4=12$
$\frac{1}{1}\times\frac{n}{1}\sim\frac{n}{1}$	corresponds to	$1\times n=n$
$\frac{n}{1}+\frac{0}{1}\sim\frac{n}{1}$	corresponds to	$n+0=n$

Approaching fractions by mixing the operations with natural numbers and fractions, a practice which is not mathematically defensible, deprives children of an opportunity to learn that among all the fractions there is a subset that "behaves" just

[5] We can go beyond the above definitions by calling such symbols as $\frac{a}{b}$ fraction forms. They can only be called fractions by courtesy of an ellipsis.

like the naturals under the operations of addition and multiplication. This is an important mathematical principle.

As a second illustration of mathematically bad practices we cite the common argument that $\frac{1}{4}\times 3$ is the same as $\frac{3}{4}\times 1$. Now, obviously, $\frac{1}{4}\times 3$ yields the same number $(\frac{3}{4})$ as $\frac{3}{4}\times 1$, but we must remember that the reason that it does yield the same result is due to a property of multiplication of pairs of numbers. The usual explanation of the equality of these two products breaks down the moment one keeps faith with the children, that is, if the discussion remains in the realm of number pairs. If this restriction is observed, then we must consider the two products in the following forms:

$$\frac{3}{4}\times\frac{1}{1} \quad \text{and} \quad \frac{1}{4}\times\frac{3}{1}.$$

These are products formed by means of two different number pairs. The usual explanations are immediately revealed as a pattern of nonsense when phrased in terms of pairs.

A more defensible approach to the whole topic of fractions is to tell the children that we are going to work with pairs of numbers and that these pairs are called fractions. *All of the work is then done in terms of pairs of numbers* and, as an important sidelight, the children are taught that a certain parallel exists between a subset of these pairs and the natural numbers. This parallelism enables us to shorten the work with fractions in many instances.

Now that we have the ideas of rate pairs and fractions, let us try using them to solve a few illustrative problems.

Problem Situations

a. Three boxes, each with the same number of apples, contain 21 apples. At this rate, how many boxes will it take to pack 84 apples?

This problem does not fit the fraction idea. It is a rate-pair situation, however:

$$3/21 \sim N/84.$$

b. Team A won one out of five games before Christmas and four out of seven games after Christmas. What is their record for the season? Their record is five out of twelve games.

Here we use another rule for addition as illustrated below:

$$1/4+4/7 \sim 5/12.$$

If we were to add 1/4 and 4/7 according to the fraction rule, we would obtain the following result:

$$\frac{1}{4}+\frac{4}{7}\sim\frac{7+16}{28}=\frac{23}{28}.$$

This result is perfectly ridiculous if it is to be interpreted as games won.

Note that in this instance we may not replace 1 out of 4 by its equivalent rate 2 out of 8. If we did, the team's record would be 6 out of 15, and 6 out of 15 is not equivalent to 5 out of 12. Hence, when using these pairs, we may not use a substitution rule as we may with fractions.

c. Jim's father spent one-fourth of his annual income for rent and two-fifths of his income for food. What part of his income did he spend for both rent and food?

This is a fraction situation and we use the rules for fractions.

Fractions and Rational Numbers

We see that up to this point fractions are pairs of natural numbers that obey the rules listed in Table I. We also see that fractions are members of equivalence sets which can be constructed, if we know one member, through the use of the rules for set membership, namely, $\frac{a}{d}$ and $\frac{c}{d}$ are members of the same set if and only if $ad=bc$. So far we have not given this set of equivalent fractions a name. It may be well to have a specific illustration in mind before giving names to sets of equivalent fractions.

Consider the fraction $\frac{2}{3}$. The rules for set membership enable us to write the following set of equivalent fractions:

$$\left\{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \frac{10}{15}, \ldots \frac{2n}{3n}, \ldots\right\}.$$

We will follow acceptable practices in calling this set of equivalent fractions a rational number. A rational number is a "family" name for a set of equivalent fractions. It is customary to use the same symbol for the rational number as that used for

the fraction. Thus, it is customary to write the following:

$$\text{The rational number } \frac{2}{3} = \left\{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \cdots\right\}.$$

Of course, it is also possible to write the following:

$$\frac{4}{6} = \left\{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \cdots\right\}.$$

In each instance, it is understood that the symbol $\frac{4}{6}$ or $\frac{2}{3}$ appearing on the left is a symbol for a rational number. It can be shown that the same rules govern rational numbers as govern fractions. This, however, is not an elementary topic, so we turn our attention to more elementary things.[6]

Rate Pairs and Per Cents

We have all used a common application of the idea of rate pairs, namely, per cents. It is common practice to read the symbol 15% as "15 per cent," which means 15 per 100. Thus we see that per cents are rate pairs. Let us see how the need for per cents arises out of a comparison situation.

John sold four out of five of his junior theater tickets, and Jane sold 21 out of 25 of her tickets. Who had the best sales record? We note that most certainly the problem does not imply that John had only five tickets to sell, and that Jane had 25 tickets to sell. In fact, the logical conclusion is that we do not know how many tickets John and Jane had to sell. It may be that John had more tickets to sell than Jane had. However, it is possible to make a comparison of the two rate pairs if we obtain a common second component for each pair. It is standard practice to make this second component 100. Hence, John's "four out of five" is equivalent to 80 out of 100, and Jane's "21 out of 25" is equivalent to 84 out of 100. It is now obvious that Jane has the best record.

The familiar three cases of per cent can be used to show how rate pairs can be used to solve each type.

Case I. In School A, 75% of the children contributed to the Community Chest drive one week before the final deadline. There are 480 children in School A. How many children contributed to the Community Chest drive one week before the deadline?

$$75/100 \sim N/480$$

(75 per 100 is "how many" per 480?)

$$75 \times 480 = 100 \times N$$

$$N = 360$$

Case II. In School A, 450 of the 750 children in the school have contributed to the Community Chest drive. What per cent of the children have contributed to the drive?

$$450/750 \sim N/100$$

(450 per 750 is the same as "how many" per 100?)

$$450 \times 100 = 750 \times N$$

$$N = 46\tfrac{2}{3}$$

Hence, $46\frac{2}{3}$% of the children have contributed.

Case III. In School A, 75% of the children contributed to the Community Chest drive one week before the final deadline. This means that 240 children have contributed to the drive. How many children attend School A?

$$75/100 \sim 240/N$$

(75 per 100 is 240 per "how many"?)

$$75 \times N = 240 \times 100$$

$$N = 320$$

Per Cents and Numbers

A per cent is not a fraction according to the way rate pairs and fractions have been developed in this paper. In fact, we do not write 5% = .05. Let us see how the habit of writing such expressions as 5% = .05 arose.

We will do this by considering another per cent situation:

Mr. Jones spent 35% of his October pay check to pay a doctor bill. If Mr. Jones' October pay check amounted to $560, how much was the doctor bill?

Using the idea of a rate pair, we can work this problem in the following way:

[6] See Edmund Landau, *Foundations of Analysis* (New York: Chelsea Publishing Co., 1951), for the complete development of the real numbers along these lines. Developing a continuous instructional program for the real numbers in school mathematics is one of the two hard problems in curriculum construction.

$$35/100 \sim N/560$$

$$35 \times 560 = 100 \times N$$

$$\frac{35 \times 560}{100} = N$$

We can use three methods to complete the work.

Method I:

$$\frac{35 \times 560}{100} = \frac{19600}{100} = 196$$

Method II:

$$\frac{35 \times 560}{100} = 35 \times \frac{560}{100} = 35 \times 5.6 = 196$$

Method III:

$$\frac{35 \times 560}{100} = \frac{35}{100} \times 560 = .35 \times 560 = 196$$

Up to this point in the paper we have implicitly used Method I to process the numbers after having an equation of the type $35 \times 560 = 100 \times N$. However, after having worked problems of this type for some time, our number sense tells us that it would be possible to develop two shortcuts.

Shortcut I: Always divide the base (560) by 100 and multiply by the "rate" number. This corresponds to Method II shown above. Most certainly, if we wanted to teach arithmetic by the rule method, it would be possible to use Shortcut I.

Example: Find 5% of 300.
5% of $300 = 5 \times 3 = 15$.

Those who are quite familiar with number ideas will find this a concise, although quite unfamiliar, way to work this type of problem. However, if this is given as a rule for solving such problems, children would find it quite unsatisfactory.

Shortcut II: Always divide the "rate" number by 100 and multiply by the base. This corresponds to Method III.

Example: Find 5% of 300.
5% of $300 = .05 \times 300 = 15$.

This method is also concise if you are relatively sophisticated about the use of number. It is no better or no worse than Shortcut I. Both shortcuts could be taught in the elementary school. For some reason, Shortcut II is commonly accepted. However, if taught as a rule, children will find this an unsatisfactory procedure.

Some persons wish to define 5% as equal to .05 and then tell children to use Shortcut II. This produces the correct answer in all cases, but it fails to develop basic understandings and a familiarity with basic number ideas.

Let us return to the original equation of rate pairs obtained from the problem about Mr. Jones' doctor bill.

$$35/100 \sim N/560$$

$35 \times 560 = 100 \times N$ (because of our cross-product test)

At this point in the solution, we have reduced a condition on rate pairs to one involving natural numbers and the operation of multiplication. This should be familiar territory for children. They now work with natural numbers to find a replacement for N. As they become familiar with the process, they will see that there are various ways to reduce the labor. Eventually, they may use either Shortcut I or Shortcut II.

Now we turn to a brief statement about using these ideas in the construction of a curriculum.

Curricular Implications

Rate pairs and fractions must be integral parts of the curriculum in the elementary school. Of course, fractions, with certain minor exceptions, are now taught beginning in the fifth grade. However, a good case can be constructed to support the introduction of rate pairs before beginning a serious study of fractions.

In the first place, the physical situations from which the child must abstract the idea of rate pairs are simpler than those from which he must abstract the idea of a fraction. To share a number of things on a 2-for-3 basis is a very simple idea. To cut a pie into three equal parts and then consider two of these parts is comparatively

complex. Anyone who has taught children can attest to the difficulty a child has in cutting a pie or a string into three equal parts.

In the second place, some of the basic properties of fractions can be developed on the basis of the child's experience with rate pairs. To have these ideas fixed before attempting a serious study of fractions supplies an excellent readiness program for fractions.

In the third place, experience has shown that the rate pair idea is feasible for the fourth-grade child. It is true that the experience has been limited to thirty or forty teachers in four or five systems,[7] but what would be gained insofar as a feasibility test is concerned if the experience were extended to tens of schools and hundreds of teachers at a cost of hundreds of thousands of dollars? Nothing.

Among the many things that teachers must do to bring twentieth-century arithmetic into the curriculum of the schools of the twentieth century, one is the introduction of the study of rate pairs and their properties.

[7] For a research study supporting informal experiences, see Della L. McMahon, "A Comparison of Two Methods of Teaching Per Cent" (Unpublished Doctor's thesis, University of Missouri, Columbia, Missouri) and Ina Mae Silvey, "The Use of Ratio or Fractional Thinking to Explore Collections in Elementary Arithmetic," (Unpublished Master's thesis, Iowa State Teachers College, Cedar Falls, Iowa).

EDITOR'S NOTE: As usual, Dr. Van Engen has raised some critical issues, given a scholarly answer to the questions, and demolished some notions heretofore cherished to varying degrees by most of us.

Lest any new reader think that Dr. Van Engen intends to teach children thus formally, let us remind you that he is here addressing adult teachers and that he assumes some sophistication with respect to symbols. He would be the first to get down to sticks and stones with children and thus objectify the ideas under consideration in so far as it is possible to objectify *ideas*.

Anyone who has tried rate language in the lower grades or the rate solution to percentage will certainly agree that it is successful and ought to be developed thoroughly. Furthermore, it can be developed early in the elementary school mathematics curriculum.

Will You Contribute to a Forthcoming Yearbook?

The Board of Directors of The National Council of Teachers of Mathematics at the April, 1960 meeting approved the preparation and publication of a yearbook to be devoted to the problem of the mathematical education of the talented student, K–12.

It is intended that the yearbook be a *source-book* of topics and materials which have been found useful in enriching the mathematics program of talented students, but which are not parts of either traditional or experimental courses.

Under the chairmanship of Julius H. Hlavaty, the editorial committee, some members of which are listed below, has accepted the responsibility of preparing the grade-level sections as indicated here. If you have material that you believe would make a contribution toward achieving the purpose of the yearbook, send your contribution to a committee member or the chairman.

K–8	VINCENT J. GLENNON Director, Arithmetic Center Syracuse University Syracuse 10, New York
7–10	JOSEPH L. PAYNE 3019 University School University of Michigan Ann Arbor, Michigan
9–11	HENRY SYER Kent School Kent, Connecticut
12, Honors	HARRY D. RUDERMAN Hunter College High School 930 Lexington Avenue New York 31, New York

Please submit your contribution no later than February, 1961. Where grade levels listed here overlap, send the material to either committee member, but not to both. Full credit will be given to each contributor whose material is used.

JULIUS H. HLAVATY
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475 Riverside Drive
New York 27, New York

Why Not Individualize Arithmetic?

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DURING THE PAST fifteen or twenty years, teaching methods in reading have developed which make adjustments to the students' abilities in all phases of learning to read. It is quite common in one elementary classroom to find two to four groups of children reading from different books and using different materials. Each group of children form a cluster of similar achievements and abilities. Within each of these clusters there will also be a noticeable range in ability and achievement.

By comparison, the arithmetic curriculum has followed a grade-level organization. Often all children are using the same text. In extreme cases, all may be found doing the same work at the same time. Any class of upper-grade elementary children is likely to have a three- to six-year range in arithmetic ability and achievement. In the grade-level teaching situation, the average learner moves through school mastering the required material step by step. This child learns rules and skills with some degree of understanding. Concepts are learned and used in the textbook situation without opportunities for experimentation. These average children have large differences in ability, but they are kept in single-ability groups as far as teaching is concerned.

The low achievers are continually frustrated and confused in this situation. They need a slow pace with repeated individual instruction and many experiences with audio-visual materials. If they pass through each grade, the achievement gap separating them from their classmates widens. The low achievers lose more and more insight until their arithmetic disability is permanent and irreparable.

The above-average child may offer the most severe problem when single-ability grouping is used. This child learns all the basic skills and uses them accurately in much less time than the regular program demands. Early in the grades he wants to learn and absorb new knowledge. If he finds his desires frustrated and no new material presented, resentment may build up. Resenting the lack of new ideas and concepts, he also may become a permanent disability in the school program.

What are some of the factors which have contributed to this situation? Many textbooks are directed toward a specific grade level. Each book is a lock-step presentation of concepts and processes. The tradition of introducing certain skills and concepts in a prescribed order has kept the curriculum grade-centered in its organization. Teachers have not had the guidance or materials to break away from this grade-level method. The noticeable gap between achievement scores and the mental-age grade placement on most standardized tests indicates that the lock-step system does not provide each child with the opportunity for maximum learning. This gap becomes even more acute as ability increases. The system is, in essence, forcing all children to the middle, or average, ability area. As a result of grade-level teaching, too high a percentage of our children are receiving a mediocre arithmetic education in our elementary schools.

The problems resulting from the lock-step presentation or even from modified group teaching demand that we do something positive. There is no one answer to these problems. In most situations each teacher must experiment with the available materials to meet individual differences.

There are plans being used which offer some answers. Among these are: grouping, latitudinal and longitudinal enrichment, special interest groups, and individual arith-

metic teaching. Let us have a closer look at these attempts to give a better arithmetic program.

Arithmetic grouping may be planned using the concepts that are the basis of grouping in reading. Achievement scores in comparison with mental-age grade placement will indicate a breakdown in the group of children. These differences are less obvious in the primary grades than in grades three to six. As a beginning, the teacher might use two groups. One group could include the lower third and the other the remainder of the class. This will remove the slow learners from the normally paced program.

When the program has operated for a period of time, the individual differences become more acute in terms of the amount of time it takes to complete the work, how fast mastery is achieved, and maintenance of general interest in arithmetic. In the work of fifteen to eighteen children, wide differences become apparent. It is soon obvious that the larger group will not run smoothly because the top third of the class learns faster and more thoroughly. This third should be the basis of another group within the class organization. These three groups will operate successfully, even though some children in each group will not gain the maximum from the learning situation.

Current pressure for skill in mathematical concepts requires that all children have the opportunity to learn a great deal more than our present curriculum offers. Gifted children in the normal classroom need some kind of acceleration or expansion into the depths of mathematical understandings. One way to help these children is to create special-interest groups at any given grade level. The groups might center upon music, drama, speech, and mathematics. Such a program will usually select those children in the class who have a basic interest in numbers. In a free-learning situation, these children can do a sort of adventuring in number and its usage. Such areas as simple geometry, beginning algebra, or social application may be suitable for discussion. Such opportunities give these capable children the chance to explore many areas which are not covered by texts in the conventional curriculum.

Special-interest groups may meet during the school day for a sixty-minute period or may be planned as an extension of the school day once a week. The average elementary teacher can give some direction to this type of program without special training in mathematics—provided the teacher really wants to create interest. This type of program augments the daily classroom program. It can give the academically talented child a really challenging experience. A free period, without restrictions, for these gifted children to think and question gives the learning situation new vitality.

All too often the stock answer to this basic question of what to do to produce better mathematics or arithmetic students is "to provide enrichment." Enrichment may only be a quick answer to an involved problem. Much enrichment, when carefully analyzed, is repetition and busy work hidden by new words in the social-usage areas. Enrichment should mean a type of learning which gives greater understanding rather than simply more practice in the processes.

In light of this definition, a child needs to know all of his whole number processes, fractional processes, and decimal processes as a basis for operation. Once all of these ideas are mastered, the horizon of experimentation and enrichment is greatly widened. Specifically, once these basics are acquired, the entire field of aviation arithmetic may be explored and studied. An example of this type of material is *The Arithmetic of Flying*, published by the National Aviation Education Council. Without these subject areas, much enrichment is only practice of already mastered material.

It must be remembered that the children who benefit from this type of learning are those who learn arithmetic skills quickly and efficiently in both depth and use. Therefore, any effort to increase the material which is not covered in the lock-step method will give a basis for this type of enrichment. These methods should probably include

the upgrading of material by one-half to one year, provided adequate controls guarantee thorough understanding. When considered in the entire school program, these concepts would allow the junior high school to carry out an intense review in the early seventh grade followed by an early introduction to algebra and geometry. Many criticisms of this suggestion are based upon the fact that so many children arrive in secondary school without the ability to use basic concepts adequately. It is possible that many of these pupils have been bored into apathy toward arithmetic ideas and skills. The mortality rate may very well be lowered if throughout the entire system we create an opportunity of freedom beyond the established formal content.

Another solution which has had some recent attempts and experimentation is the individualized teaching program. This concept takes teaching to the ultimate in meeting individual differences. Each student is taught and learns at his level of achievement, regardless of ability. Initially this idea is rejected by many because it sounds physically impossible or because it smacks of too many "modern methods." It is not impossible when properly organized and implemented. It is decidedly nontraditional in approach. However, it makes the maximum use of the psychological basis of learning and at the same time avoids the traps of group teaching already indicated. Certainly the key to this type of instruction centers upon the teacher's wanting to give each child the most education possible. The author has put into operation such a program in a sixth-grade class in Washington School, Culver City Unified School District, California.

The program is based upon having a range of textbook materials in the room. This does not mean more copies of one book but greater variety within the same number of textbooks. Each child is placed at the point in the texts at which he achieves best. The child meets individually with the teacher and learns the required material. The assignment is given, and the child works the required problems. He checks his own work and writes a description of any error. Upon completion, he signs an appointment list. The teacher holds conferences in the order in which the children finish their work. If any student cannot do the work or cannot understand, he raises his hand. The teacher acknowledges this, and gives him help before the next child on the appointment list is met. A check list is maintained for all the skills so that nothing is overlooked. The assignments are moved from text to text to take advantage of the best ideas from each author. The arithmetic period is usually about sixty minutes, and, as a rule, each child is seen every other day.

All of this may seem complex and involved, but the results in terms of the children's attitude and class control are gratifying. The slow learner is not compelled to meet standards of the group and relaxes to learn more easily what he can. The average learner can understand as much as he wants above the required basics of the traditional grade-level situation. The bright child has the opportunity to expand his thinking in any area he desires and is not held to the slower pace of the other learners in his arithmetic group.

Some of the details of this program can be altered to meet various conditions resulting from the grade level and materials, but the basic idea gives each child the maximum he can use in the arithmetic program.

At the outset of this discussion, the problem presented was that the elementary reading program has been geared extensively to the theory of individual differences whereas the arithmetic program has made only moderate adjustments to the same theory. These attempts have not been carried out in a consistent way, nor has their importance been emphasized in the daily teaching program. Some effort must be made to plan an arithmetic curriculum which meets the needs of all the children in the normal elementary classroom. Certainly these attempts cannot be on too great a scale. The usual materials and books will need careful evaluation. Teachers will have to begin to

think about arithmetic in a new perspective. As educators dedicated to giving every child the best education, we must eventually adjust the total curriculum so that each child will gain the *maximum* in terms of his ability and interests. So—why not individualize arithmetic?

EDITOR'S NOTE: There is much in the wind now about releasing able children to learn what they are capable of learning rather than the small measured doses suitable for a hypothetical average child. This article summarizes some techniques which should be effective. It is to be hoped that we will soon have research data with which to judge the effectiveness of such practices.

Are You Planning to Submit a Manuscript?

We invite you to write for publication in THE ARITHMETIC TEACHER. We ask, however, that you prepare your manuscript according to the following instructions:

1. All manuscripts should be typewritten, double- or triple-spaced, on 8½ by 11 good quality paper, with *at least* one-inch margins on both sides.

2. References and footnotes should be numbered consecutively and placed at bottom of the page.

3. Prepare footnote references as follows:

To a book:

[1] Francis J. Mueller, *Arithmetic, Its Structure and Concepts* (Englewood Cliffs, N. J.: Prentice-Hall, Inc., 1956), p. 17.

To an article in a periodical:

[2] Marshall H. Stone, "Fundamental Issues in the Teaching of Elementary School Mathematics," THE ARITHMETIC TEACHER, VI (October, 1959) p. 179.

To a yearbook:

[3] Robert L. Swain, "Modern Mathematics and School Arithmetic," in *Instruction in Arithmetic*, Twenty-fifth Yearbook of The National Council of Teachers of Mathematics (Washington, D.C.: The National Council, 1960).

To a technical bulletin, pamphlet, or similar publication:

[4] *Studies in Mathematics Education* (Chicago: Scott, Foresman and Company, 1960), p. 22.

4. If a bibliography is included at the end of the article, use the following guide.

For a book:

SPITZER, HERBERT F. *The Teaching of Arithmetic.* Boston: Houghton, Mifflin Company, 1954.

For an article in a periodical:

PETERS, ANN C. "The Number System and the Teacher," THE ARITHMETIC TEACHER, IV (October, 1957), 155–60.

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THE ARITHMETIC TEACHER
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Enrichment for Understanding

PATRICIA SPROSS
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IF WE ARE to make room for much that is new and necessary in today's crowded curriculum, it would seem that we must either give up some of the subjects now offered or compress the ones we have into shortened periods of instructional time. World developments and cultural pressures make it necessary to find a way to teach more effectively and somehow induce youngsters to learn more material faster. Often because of lack of time and adequate facilities, we are unable to practice the methodology current educational research might indicate advisable. This has often been the case with homogeneous grouping. Time schedules and facilities are not always flexible enough to permit our taking advantage of its possibilities. This has been particularly true in our four fifth- and sixth-grade classes in Lansing, Michigan, which are taking part in the Study for the Use of Special Teachers in Science and Mathematics.

Since our classes are limited to 35 minutes per day for both instruction and "seatwork," we felt that a good understanding of arithmetic would have to replace so-called drill materials. There would be little time for grouping or practicing skills. Our purpose would be to find out, if possible, whether understanding might not result in as effective learning as does repetitive drill; if such an understanding might not lead to proficiency in the manipulation of mathematical symbols; and if the concepts involved in understanding a mathematical process could really be taught to children of all abilities in only 35 minutes a day.

Our four classes are composed of typical fifth- and sixth-graders, representing all levels of ability found in the usual classroom. Any grouping would result in extremely short periods of instruction for each group. Still, the question plagued us all as to whether the theory that should be taught in connection with the understandings might not be too difficult for those children "who had always needed extra help." So we tried to take the content dealing with mathematical theory and literally stretch it to suit all the levels of ability found in our classes.

At the outset, it was evident that all these children needed to be able to *do* something with the concepts that they learned. Something needed to be found to replace seatwork materials. It has been by giving all children something to do with what they have learned that we have been able to fit the content to their various abilities.

Finding a Use for a Number System

On our first day of class we asked all the students to forget that they could count (enumerate), that they knew any symbols to represent quantity, and even that they knew any words for this. Our problem then became one of describing quantity without using any of past learnings in the arithmetic processes. We discussed ways of doing this. We investigated selected historical references to find how mathematical systems might have developed as a means of "keeping track of something": the way early man might have kept track of his flocks, one-to-one relationships, and so on. After each investigation, we discussed the convenience and suitability of the method. Each child invented a mathematical system of his own to symbolize quantity and tried to work simple problems in it.

It did not take the youngsters long to discover such things as: keeping track of your sheep on the wall of a cave was impractical because when you moved "you couldn't take your accounts with you"; also that a carving in stone was "impossible to erase if you made a mistake"; if you had a

very large flock, a one-to-one relationship was "foolish" because it would take too many pebbles to keep your books, and if you spilled the pebbles, "you wouldn't know where you were."

After discussions like these, each child revised and worked in his own number system. We made more discoveries. Some children who did not take the project seriously at first found that their number systems were too intricate. They also discovered that the symbols were too difficult to memorize and so were not very useful. Some slow children believed that it was a waste of time to invent symbols counting to ten, five was better and easier to memorize! In a few instances they investigated place-value concepts, but for the most part the children seemed to forget about zero. When they were investigating the Egyptian number system no one even mentioned the fact that this system did not have a zero.

At the end of about four weeks, we evaluated our experience. It might be said that these children had not been *taught* a single thing, but they had found out a great deal. They had not only devised a number system of their own, but they had compared it to Egyptian, Mayan, Roman, Greek, Hindu-Arabic, and other systems. Some "slow" children had discovered that if the Egyptians had made use of subtraction "as the Romans did" they could have improved their number system. It was always noticeable that the slower children were quick to find ways of saving themselves time and effort. All levels of ability were now applying and using the terms of "suitability," "consistence," and "usefulness" in a number system. They knew how multiplication, subtraction, repetition, and addition could function in numeration. They knew, not because they had been taught rules, but because they had tried it.

Motivation

So far we had done no "problems in the text." It would have been impossible to tell any anxious parent how Johnnie was doing in his texbook or that he had reviewed all the combinations. However, most of the children were very aware of the way numbers function as a useful cultural tool, and they had dealt with the combinations in constructing a number system. One day, after giving a fifth grade a lesson to do "in the text," the only question that was asked was a casual, "What number system shall we do it in?"

One important thing had been accomplished. These children needed no urging to do their arithmetic. The problem began to be how to get them to stop in time to start their next class. Everyone in the arithmetic classes was doing something *at his own level*, but in the area of our investigation. Each lesson contained material at all levels, but drawn from a particular concept. Understanding was always our first consideration. Some children did many problems, others labored over just a few for the whole time. The investigations of talented children took them into the fields of algebra, geometry, mathematical history, and textbooks on theory. Our biggest problem was to keep these girls and boys supplied with materials. Often more talented students would be working with those of less ability. As we studied each mathematical process, our main goal was to find out all that we could about it. No one was required to work any given assignment. He was asked only to produce something showing what he had found out and to be able to explain it to others.

Projects at All Levels

When we returned to the text, we covered the material in record time, using the understandings that resulted from our investigations. We continued the rest of the year in the same manner. No one was left out of any part of our study because of lack of ability. Each child contributed something to the total learning situation that was uniquely his own. Typical were the projects resulting from our study of numeration ("keeping track of things"). Some children gathered pebbles and showed a one-to-one relationship, others 1-to-10 or 1-to-50, and

gave their reasons for doing it. We collected different number systems and studied their history and development. Some children did extensive research while others only found out about the base of a system. Children who had been very slow in reading were adding words to their vocabulary that would most certainly have been considered too difficult for them, and they discussed the function of a number system as casually as the ball game played at recess.

Fractions from Measures

After our work with numeration, we spent time studying measures of length, area, and volume. Using actual measuring devices, the children developed their own charts of relationships between the fractional parts of various measures. With great enthusiasm they developed tables such as:

$$4 \text{ in.} = \tfrac{1}{3} \text{ ft.} \quad \text{or} \quad 3 \text{ ft.} = 1 \text{ yd.}$$

$$\therefore \ 2 \text{ in.} = \frac{\frac{1}{3}}{2} \text{ ft.} \qquad 1\tfrac{1}{2} \text{ ft.} = \tfrac{1}{2} \text{ yd.}$$

$$2 \text{ in.} = \tfrac{1}{6} \text{ ft.} \qquad \frac{1\frac{1}{2}}{2} \text{ ft.} = \tfrac{1}{4} \text{ yd.}$$

$$\therefore \ \frac{\frac{1}{3}}{2} \text{ ft.} = \tfrac{1}{6} \text{ ft.}$$

They explained these tables by careful thought and reasoning—not with any particular knowledge of the symbols or processes involved. They reasoned that $1\frac{1}{2}$ feet divided by 2 was equal to 6 inches plus 3 inches, or 9 inches; and that 4 times 9 inches would give them 36 inches, so they concluded that they were correct in their assumptions. Their objective was always to make "true statements" concerning the measures by using what they knew to be true.

From this line of thought they developed such "true statements" as:

$$\frac{1\frac{1}{2}}{3} + \frac{1\frac{1}{2}}{3} = \frac{3}{3} = 1$$

$$\frac{2\frac{1}{2}}{4} = \frac{5}{8}$$

They were reasoning that: "because you have two-fourths and that is equal to four-eighths, plus a half of one-fourth, which is one-eighth . . . then you must have five-eighths."

The children still had done nothing with lowest common denominators or fractions. They used only measures in the room which they could manipulate. However, after these investigations, fractions seemed to be a "natural." Again they worked from what they knew to make true statements. They made their own rules to help them with fractions. Typical of these were the two following examples:

$$\frac{n}{n} = 1 \quad \text{and} \quad \frac{n}{n} + n_1 > 1$$

Whenever they were in need of a symbol to write their rules we helped them find one, presenting it as a convenient way of writing something—not as an "advanced mathematical" symbol.

Exponents and the Powers of Ten

Big numbers were no longer frightening when children found out how simple exponents were. In studying place value, they worked out the uses of exponents in expressing powers of ten. Starting with $10 \times 10 = 100$, pupils learned that this is expressed as 10^2. Knowing this, they were able to express 1000; 10,000; 10; and 1 as powers of ten. They discovered from this that we subtract exponential numbers to divide and add them to multiply. They took it all very casually and were never aware that this process was anything but just a convenient way of manipulating large numbers. Some children could see that if 100 was expressed as 10^2 and 1000 was expressed 10^3, then 500 could be expressed with an exponent that would be somewhere between 2 and 3. By themselves, some of the talented students could thus imagine the use of logarithmic expressions. However, for the class as a whole, the use of exponents was confined to helping them understand the meaning of place value.

Concluding Activities

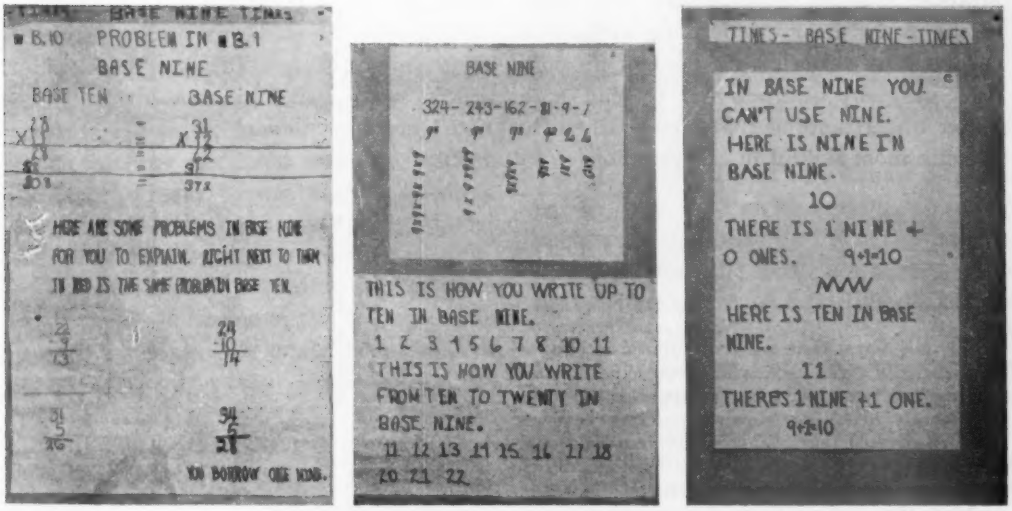
The youngsters learned many ways to do the various arithmetical processes. Each child was always free to use any method that best suited him. This enabled us to circumvent the difficulty presented when someone said, "But my mother says that she didn't do it that way when she went to school." We just learned mother's way and added it to our collection.

At the end of the year we got all our projects and materials out of the cupboards at Fairview School and, with the special science teacher, presented a Science and Math Fair to the patrons of the school. Each child had taken it upon himself to produce something that would show others a part of what we had learned in our arithmetic class. Even slow children, who had now a firm foundation in numeration, had no difficulty in producing a counting chart in another base of their own choosing. More talented youngsters made multiplication and addition tables in other number bases or compared base 10 with some other base.

The children had planned costs of family vacations, holiday meals (down to the ast damask table napkin), family clothing, and house furnishings. One talented girl in the sixth grade designed a house, drew the floor plan to scale, made a model of it, and figured the cost of construction. A mother, who let her daughter help with the family budget, shared the figures with us. We even knew how much it cost to feed the family dog per day.

The only difficulty we had was in trying to get the children to realize that they probably knew more about theory of arithmetic than many of the adults to whom they would be talking at their fair. Therefore, they would need to make careful and detailed explanations about their work. Some of the children made tables in other number bases and passed them out to parents. They made charts showing: problems solved in other bases, the same problem solved in many ways, the history of the development of measurement, how different ways of making number symbols changed, a comparison of Fahrenheit and centigrade thermometers, and how to change from one scale to the other.

As we look back on the year, we feel that the most significant thing about our work is that no one was ever left out of what we


BASE NINE TIMES
PROBLEM IN BASE TEN
PROBLEM IN BASE NINE
BASE NINE
BASE TEN
BASE NINE
HERE ARE SOME PROBLEMS IN BASE NINE FOR YOU TO EXPLAIN. RIGHT NEXT TO THEM IS THE SAME PROBLEM IN BASE TEN.
YOU BORROW ONE NINE.
BASE NINE
THIS IS HOW YOU WRITE UP TO TEN IN BASE NINE.
1 2 3 4 5 6 7 8 10 11
THIS IS HOW YOU WRITE FROM TEN TO TWENTY IN BASE NINE.
11 12 13 14 15 16 17 18
20 21 22
TIMES - BASE NINE - TIMES
IN BASE NINE YOU CAN'T USE NINE.
HERE IS NINE IN BASE NINE.
10
THERE IS 1 NINE + 0 ONES. 9+1=10
HERE IS TEN IN BASE NINE.
11
THERE'S 1 NINE + 1 ONE.
9+2=10

Children's Charts Showing Numeration Systems

did because of his lack of ability; or, conversely, no one was *limited* as to what he could learn because he might be in a group that set him apart from others. We all worked together to find out all that we could about arithmetic. As a result of this positive approach, it seemed to each of us that real learning took place, interests grew, fascinating areas were discovered, and the pupil attitudes that developed were A+.

What About the Future?

We might assume from our study that slow children can investigate the same areas of interest that are usually limited to more talented youngsters, but that the production of the slow child will be less. If so, is there any just reason to shut the slow child out of such experiences so that he may keep drilling on tables? A simple understanding of arithmetical theory seemed to be a good starting place for us. We cannot be too hasty in changing our arithmetic content; nor can we expect elementary teachers with little or no background in mathematics to become proficient overnight. Neither can we expect parents brought up in the tradition of the 3 R's to respond positively to a new arithmetic teaching. Computation and skill in getting the right answer are still, and will be, a goal in arithmetic teaching. It is possible however that an acquaintance with some of the basic theory underlying arithmetic can be used to enrich the content for children of many abilities and to lead to a better understanding that enables them to become more skillful in computation.

EDITOR'S NOTE: Here is an enthusiastic account of one teacher's experience with some experimental materials. Before long, many of you will have the opportunity to try enrichment materials or even whole programs that are very different from the usual arithmetic.

We hope you will enjoy it as much and handle the problems that such a venture presents with the same aplomb shown by Mrs. Spross.

Changing from One Number System to Another

EDITOR'S NOTE: Linda Deans is an eighth-grade student at the Thomas A. Edison Junior High School, Brighton, Massachusetts. She wrote this explanation of changing from one number system to another as it was "discovered" by the eighth-grade students in her mathematics class in their study of number systems. Their teacher is Ralph A. Welling.

When changing from one number system to another number system, you don't necessarily have to go to the usual ten system before going to the number system you want. Omit going to the ten system. Then divide the number system into the number to be changed. For the divisor use the number system (such as six for the six system) and for the dividend use the number you want changed. Take the dividend of the next example from the quotient of the first example, and the divisor will be the same. You repeat this process until your quotient is zero. List all the remainders and to get your answer read the remainders from the bottom up.

To check your answer, change both numbers to the ten system. They both should be the same.

Example

423_6 to base 4

$$
\begin{array}{r}
103_6 \\
4\overline{)423_6} \\
4 \\
\hline
23 \\
20 \\
\hline
3
\end{array}
\qquad
\begin{array}{l}
3 \\
3 \\
1 \\
2
\end{array}
\qquad
423_6 = 2133_4
$$

$$
\begin{array}{r}
13_6 \\
4\overline{)103_6} \\
4 \\
\hline
23 \\
20 \\
\hline
3
\end{array}
$$

Check

$$
\begin{array}{rl}
3\times\ 1 = & 3 \\
2\times\ 6 = & 12 \\
4\times 36 = & 144 \\
\hline
 & 159
\end{array}
$$

$$
\begin{array}{r}
2 \\
4\overline{)13_6} \\
12 \\
\hline
1
\end{array}
$$

$$
\begin{array}{rl}
3\times\ 1 = & 3 \\
3\times\ 4 = & 12 \\
1\times 16 = & 16 \\
2\times 64 = & 128 \\
\hline
 & 159
\end{array}
$$

$$
\begin{array}{r}
0 \\
4\overline{)2} \\
0 \\
\hline
2
\end{array}
$$

An Experiment in Arithmetic Acceleration

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The Problem

TWO YEARS AGO, in the annual review of achievement test results, our school district became concerned with arithmetic accomplishments. The greater number of our rapid learners—thought to be two and three years above grade level—were not achieving up to expectancy in this area. They were doing so in reading and language. A review of national norms shown by the California Testing Bureau[1] points this out as a national problem. The range of scores in arithmetic is narrower and scores cluster toward the middle of the group. Our school district was no exception in this problem. All groups concerned—teachers, parents, administrators, and school board—felt that we should attempt to bring the rapid learners up to expectancy in arithmetic.

Procedure

The teaching staff worked on this problem throughout the 1958–59 school year. The efforts made toward improving arithmetic instruction included short course work for teachers, making additional concrete teaching aids, and visits by teachers to other schools. These things, however, had been done before in our district, though not in so concentrated a period.

The entirely new procedure applied to the problem was to remove the ceiling on the use of materials. We educate to the sixth grade, and we had previously used basic texts only to that level, with some additional enrichment books. We brought in basic texts through the eighth-grade level and decided to launch on a program of vertical enrichment limited only by the students' ability and the time factor. The degree of each child's attainment would be recorded and kept in his permanent record. This was the permanent record procedure already being used in reading. The teacher would then know where to begin the following year.

The upper grades, four through six, began the year with a form of individualized arithmetic. The children soon found their levels, and we worked the major part of the year as groups.

Results

One group of five very talented sixth-graders, IQ's ranging from 125 to 171, completed the seventh-grade text. The majority of the students ended the year over halfway through next year's material. These children's median intelligence rating was 110. A small group of children with median intelligence at 96 were just able to finish their grade-level material.

Scattergrams I and II compare September to May. A student working above expectancy would be to the left of the diagonal that runs from the lower left corner to the upper right corner. A student's position on such a scattergram relative to expectancy would not change from year to year if he continued consistent growth. A comparison of the two scattergrams shows a change for the group from slightly below expectancy to slightly above. The students below in October numbered twenty, while four were

[1] Ernest W. Tiegs and Willis W. Clark, *Manual to California Achievement Test, Elem., Form WXYZ*, (1957 ed.; Los Angeles: California Testing Bureau), p. 54.

above. In May, there were eleven below and nineteen above. Both scattergrams show a cluster near expectancy. The May scattergram shows a group of six sixth-graders above 9.5 grade placement. Though high achievers, they were not all up to expectancy, even after as much as three years' gain. This group attained a median growth of 2.25 years. The median growth for all children participating in the study was 1.53 years.

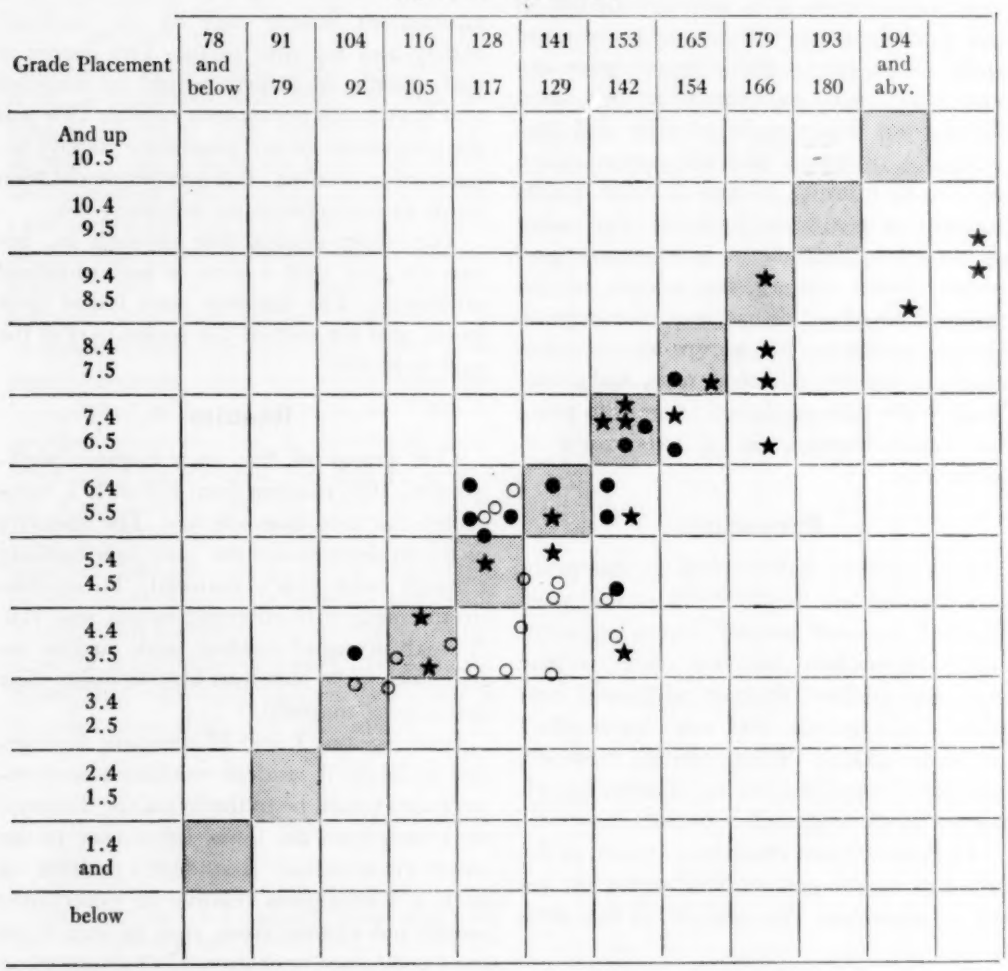
Though much of the emphasis was on the rapid learner, it is important to notice that there were more over-achievers in the least able group.

No emotional problems developed from this experiment. Teachers observed an increased interest in the subject.

SCATTERGRAM I

October 1958

(IQ×Chronological Age)

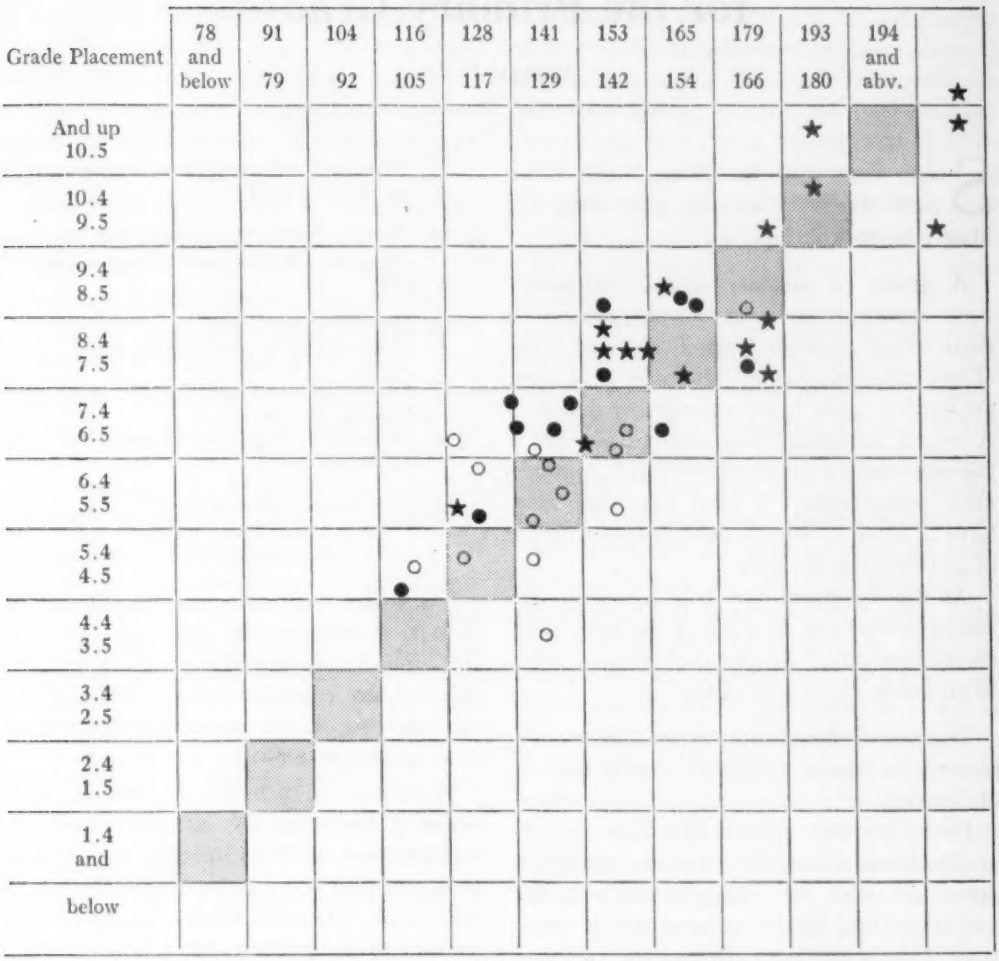


	Below	Average	Above
○ Fourth-Graders	7	6	1
● Fifth-Graders	5	6	3
★ Sixth-Graders	8	8	

SCATTERGRAM II

May 1959

(IQ×Chronological Age)



	Below	Average	Above
○ Fourth-Graders	3	7	3
● Fifth-Graders	2	3	10
★ Sixth-Graders	6	4	6

Recommendations

The results of this experiment lead us to recommend these definite procedures for similar groups.

1. Have ample advanced material available so that children can proceed according to ability.

2. Work in groups and move along as rapidly as groups are able.

3. Record on permanent records the degree of text achievement at the end of the year for use the following year.

EDITOR'S NOTE: This author offers more data to support the view that students profit by being liberated from lock-step education. Interestingly enough, it appeared that the rapid learners weren't the only ones who profited from this liberalized program. Attitudes toward learning are contagious, and children learn much from each other. It isn't too surprising that the below-average pupils found this liberation stimulating too.

Enrichment in Arithmetic for the Primary Grades

MORRIS PINCUS
Public School 194, Brooklyn, New York

SOME TIME AGO the New York *Times* carried a story which I am taking the liberty to paraphrase.

A group of primary school youngsters were enjoying recess on the playground of their school opposite Cape Canaveral. One of the youngsters said, "It looks as if they are getting ready to launch another missile." A second lad said, "Yes, and it looks like a big one—over a hundred feet tall." Another child commented, "I hear that they now have a thrust of 40,000 pounds and a speed of 23,000 miles per hour."

At this moment, the bell signaling the end of recess rang, and one of the boys commented wearily, "Aw, shucks! Now we have to go inside and count beads."

The story, though apocryphal, succinctly makes the point which I would like to elaborate.

In my frequent visits to first- and second-grade classes during the past few months, I often talk with the children, individually and in groups. Here are some of the questions I injected into discussions and the answers I received.

1. Q: You said Enos Slaughter was born in 1916. How old is he now in 1960?
   A: 44 years.

   Q: How do you know?
   A: I know 16 and 40 are 56, and 4 more are 60.

2. Q: We have five pieces of cake for seven people. How can we divide it?
   A: I won't have any. If we divide one piece in half, we will have enough for six people.

3. Q: Does your house cost as much as that house?
   A: No.

   Q: Two of those houses?
   A: No.

   Q: What then?
   A: About two of our houses will equal three of those.

   Q: How much will one of your houses equal?
   A: About 1½ of them.

4. Q: You have 28¢. How much will you have when you get your new allowance (40¢)?
   A: 68¢

   Q: How did you get 68¢?
   A: 40 and 20 are 60, and 8 make 68.

5. Q: I have 86¢ and I need a dollar. How much more do I need?
   A: 86 and 10 is 96 and 2 is 98 and 2 is a dollar. You need 14¢.

6. Q: How much are 177 and 137?
   A: 100 and 100 is 200. 75 and 25 is 100. This makes 300. 2 and 12 is 14. So we have 314.

These illustrations were selected at random from many similar ones to demonstrate the type of thinking many of our primary children are capable of doing. Of course I am referring to the more able children in those grades, not to all.

Principals of neighboring schools with whom I discussed this situation indicated that primary teachers in their schools were unhappy with the present course of study in arithmetic. They felt there was not enough content to do a reasonable job with bright children.

We decided to utilize objective measures to verify these feelings. The New York Inventory of Mathematical Concepts, Level I, is given in our school system at the end of the first year or at the beginning of the second year. The Level II test is given at the end of grade two or the beginning of grade three. These tests measure children's understanding as well as their achievement in skills.

We administered the Level I test to 352 children at the beginning of grade one, and the Level II test to 362 children at the beginning of grade two. The results, although anticipated, were disturbing.

In the Level I test, 24 per cent of the children scored in the 90th percentile or higher; 15 per cent scored between the 80th and the 89th percentile; and 13 per cent scored between the 70th and 79th percentile. Thus, in October, 52 per cent of the children scored between the 70th and 99th percentile in a test which is normally given at the end of the year.

The results in the Level II test were similar. Of the children tested, 54 per cent scored in the 70th percentile or higher, with 24 per cent of these children scoring between the 90th and 99th percentile.

Now the comment might be made, "But these must be very bright children." And the answer would be, "Yes, indeed. Their average IQ was 115." However, what is the mathematics program which has been laid out for these children? Does it provide adequately for this type of child?

H. F. Spitzer, in his article "Arithmetic in Kindergarten, Grades 1 and 2" [4]* states:

> An examination of current pupils' textbooks and the accompanying teachers' editions shows that the following areas comprise the bulk of the arithmetic content in grade one:
>
> —counting from 1–10 and then to 100;
> —reading and writing of numbers, 1–10 and then to 100;
> —the cardinal meanings of the numbers 1 to 10;
> —the collection idea as applied to the numbers from 10 to 100;
> —addition and subtraction of amounts (*sums to 10* and minuends of 10 or less);
> —recognition of most commonly used coins;
> —some of the relatively simple aspects of telling time;
> —some work with such measures as the quart, cup, and pound;
> —facts with sums through 12 or 14 and the addition of two-digit numbers without carrying are included.

When we question whether this program is adequate for bright children, we are told: the above is the program for average children. It needs enrichment for bright children. However, the enrichment should be horizontal, not vertical. We must not touch the content of the grade above [5].

How much enrichment can one do within the number 5, or within the number 10 in grade one? Or within a few teen numbers in grade two? Madden says [6], "A danger of horizontal enrichment standing alone is that it becomes a disorganized experience with too high a portion of mathematical recreations and other activities that have little value in leading to a more complete understanding of mathematics. The result may be a diminished challenge for the pupil and a related loss of interest in mathematics."

The course of study in the primary grades for brighter children is so inadequate that many teachers of those grades find themselves spending ten to fifteen minutes a day, or less, on arithmetic. Even with that limited allotment, many feel there is not enough content. And lest some readers get the idea that these teachers were stressing the skills at the expense of meanings and understandings, I hasten to add that these teachers felt they were stressing meanings "ad nauseam." The teachers also made the point that they were enriching as much as they could, but it was not adequate.

Discontent with the present courses of study in arithmetic for the able child is rising in many places [1, 6, 7, 8]. Many teachers feel much more content could be taught successfully in the primary grades, even to the average child. This procedure would permit moving down content now taught in the middle grades to lower grades.

Some educators are experimenting with enriching the course of study for the primary grades by teaching geometry [2]. Others talk of "Diagonal Enrichment": dipping into the work of the grade above as well as enriching horizontally [6]. One plan provides for individual differences by grouping pupils according to ability in arithmetic, disregarding grade lines, and by replacing the grade level curriculum with a nongraded program in which all students advance along the sequence of understandings and skills at rates which are suited to their individual learning capacities [3].

(*Concluded on page 417*)

* Figures in brackets refer to items in the bibliography at the end of this article, p. 417.

Study Questions for Reviewing Arithmetic

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LEARNING STUDIES INDICATE that good summaries are a means of identifying and clarifying points not yet understood and that they also provide an excellent way to retard forgetting. Yet in general, professional books and textbooks in the field of arithmetic give little attention to the need for and possible values of summarizing work through fresh or promising practices in review. A common statement in professional books is that "Maintenance exercises on facts, terms, and ideas are in most series of books inadequate," but arithmetic textbooks continue to offer review exercises which are little more than periodic practice on ability at mechanical types of operation.

For some time review questions have been accepted as one effective technique in teaching the content subjects. In contrast, it seems safe to assume that such summary questions in arithmetic have received little emphasis. Although drill may not be considered the prime factor in learning, and though some effort has been made to "make sense" in introductory phases of new work, review appears to revolve almost entirely around the mechanical manipulation of numbers with little effort at checking meaning, providing for active discovery or rediscovery of number relationships, encouraging generalizations, or other aspects that seem logically involved in the "meaning theory."

To check the understanding of work at regular intervals, some teachers now use a procedure that involves "review study questions" as a summary device. After the period of intensive study of a particular phase of arithmetic, a summary period devoted to study and discussion of thought-provoking questions seems helpful. The following questions illustrate the type suggested in such exercises.

Grade Three: Mixed Practice

1. Can you subtract to find the answer to 24 divided by 7?
2. Can you use the number line to find the product of 7×4?
3. Can this problem be solved by using a drawing? If three girls divide 24 trading cards equally among themselves, how many cards will each girl get?
4. Mary wasn't sure about her answer for this example: 94–36 leaves 58. What two good ways of checking could she use?
5. Can you show that 3×7 is equal to 7×3?
6. In the example 701–563, how do you borrow a ten?

Grade Four: Two-Figure Multiplication

1. Is 10×50 the same as 5×50 plus 5×25 plus 5×25?
2. In this example, why do we "carry" the 3 and add it to 12?

$$\begin{array}{r} 38 \\ \times 4 \\ \hline 152 \end{array}$$

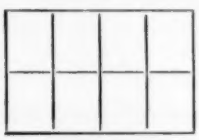
3. Why is the circled 6 placed under the 8 and 1?

$$\begin{array}{r} 86 \\ \times 14 \\ \hline 344 \\ 8\textcircled{6}0 \\ \hline 1{,}104 \end{array}$$

4. What are three ways to show that eleven 15's equal 165?
5. If 13×42 equals 182, how many 13's are contained in 182?
6. What are two good study exercises for learning the multiplication facts?

Grade Four: Common Fractions

1. Can you show by a drawing that $\frac{2}{8}$ equals $\frac{1}{4}$?
2. Is $\frac{1}{4}$ one of the equal parts of $\frac{3}{4}$?
3. If you have eaten $\frac{1}{8}$ of a cake, was the cake large enough for 7 other pieces of the same size?
4. The fraction fact $\frac{1}{4} \times 12 = 3$ goes with what division fact?
5. How can you use the rectangle to help you find how many $\frac{1}{4}$'s are equal to $\frac{3}{8}$'s?



Grade Five: Two-Figure Division

1. Can you show three ways of finding how many 20's are in 80?
2. What division fact can be used to give you a hint of the first quotient figure in this example? 23)138
3. Why are there no hundreds or tens in the quotient in this example? 30)160
4. How do we get the 138?

$$\begin{array}{r} 6\text{ r}1 \\ 23\overline{)139} \\ 138 \\ \hline 1 \end{array}$$

5. Billy did three division examples. He checked one by subtraction. Which one?

$$\text{(a) } \begin{array}{r} 24\text{ r}6 \\ 3\overline{)78} \end{array} \qquad \text{(b) } \begin{array}{r} 3\text{ r}6 \\ 24\overline{)78} \\ 72 \\ \hline 6 \end{array} \qquad \text{(c) } \begin{array}{r} 13 \\ 6\overline{)78} \\ 6 \\ \hline 18 \end{array} \qquad \begin{array}{r} 78 \\ -24 \\ \hline 54 \\ -24 \\ \hline 30 \\ -24 \\ \hline 6 \end{array}$$

6. When the divisor is 20, what is the largest remainder we can have?

Grade Five: Mixed Practice

1. What is the largest four-figure number you can write with 3, 8, 5, 4?
2. Does the multiplier or the multiplicand tell you the number of partial products?
3. How can you find the average of four numbers?
4. Which work would you do to find the value of n in $275+n=750$?

$$\text{(a) } \begin{array}{r} 750 \\ +275 \\ \hline \end{array} \qquad \text{(b) } \begin{array}{r} 750 \\ \times 275 \\ \hline \end{array} \qquad \text{(c) } \begin{array}{r} 750 \\ -275 \\ \hline \end{array}$$

5. Can you use the division example

$$\begin{array}{r} 8 \\ 75\overline{)600} \end{array}$$

to tell how many 8's are in 600?

6. Why is a "little more than 6300" a reasonable check for 72×94?

Grade Six: Addition and Subtraction of Common Fractions

1. What is the best way to change a mixed number to an improper fraction?
2. When is changing terms of a fraction to higher terms needed?
3. How is the common denominator for such fractions as $\frac{2}{3}$ and $\frac{1}{4}$ found?
4. In working an example, Bill changed $4\frac{3}{12}$ to $3\frac{1}{12}$. How?
5. If you add two fractions and get a sum of $\frac{4}{5}$, what else must you do?
6. How would you compare ages in a fractional way when you know that Mary is 12 years old and her sister is 2 years old?

Grade Six: Multiplication and Division of Common Fractions

1. Do you have to change $2\frac{2}{3}$ to an improper fraction to find the product of $2\frac{2}{3}\times5$?
2. How might the product of $\frac{2}{3}\times\frac{1}{2}$ be found with a drawing?
3. Why is the product of two proper fractions always less than 1?
4. Write in words this number question? $14\div\frac{1}{2}=n$.
5. To divide $2\frac{4}{5}$ by $1\frac{2}{5}$, is it easier to use the common denominator method or the inversion method?
6. How can you know, before dividing, that the answer to $\frac{7}{8}\div\frac{2}{3}$ is greater than 1?

Grade Six: Decimal Fractions

1. Can you explain why .2 and .20 and .200 have the same value?
2. If the decimal point were omitted from 3.64, how much larger would the number become?
3. When .455 is changed to a common fraction form, how many zeros will there be in the denominator?
4. Can you write the decimal meaning two rows of small squares and 5 small squares on the hundred board?
5. How would 6 hours 30 minutes be expressed as a decimal fraction?
6. How can you change $\frac{3}{4}$ to a decimal fraction?

Grade Six: Multiplication and Division of Decimal Fractions

1. Can you show how you would check $3\frac{3}{10}\times2\frac{3}{10}$ by use of decimals?
2. What is possibly the best thing to do before multiplying a decimal fraction and a common mixed fraction?
3. What checks can you use to find out whether the decimal point in the product $.4\times.32$ is correctly placed?
4. Which of the following would you use to find the number of six-tenths in two?

$$\text{(a) } 2\overline{).6} \qquad \text{(b) } \frac{2}{.6} \qquad \text{(c) } .2\overline{).6}$$

5. When dividing $1.2\overline{)12}$, what is the reason for rewriting the example this way $12\overline{)120}$?
6. How could you state the meaning of this example in at least two ways?

$$2\overline{).2}$$

The purpose of presenting these review study questions is to emphasize that review in arithmetic need not be a deadening repetition of the same, or nearly the same, material already covered. Repetition for those pupils who have already learned may only make boring what previously held their attention; repetition for those pupils who

failed to learn may only expose them again to the materials they do not understand.

Review in arithmetic should aim to deepen the understanding of previous learning. The process should be carried a step further, and its application should be given further refinement. In this way, review ceases to be an uninteresting rehash; rather, it is used to attain further refinement in skill and understanding based upon past learnings. With questions of this type, review becomes more than additional practice in skills and abilities; it offers an opportunity for reteaching whereby pupils have another opportunity to learn, or learn better, or learn more about certain understandings.

Such teaching is appropriate because much learning is at first partial; pupils differ in what they acquire; there is forgetting; and more mature degrees of understanding become possible as pupils mature. The drill type of review is limited because it makes little provision for a pupil to learn what he previously failed to learn or what he has learned and forgotten. It fails to increase the pupil's understanding of the process; and it fails to take advantage of very important generalizations which should be redeveloped when the topic is reviewed.

Let us consider more specific characteristics of the review study questions. Note that the questions indicate the importance of pupils having a good deal of experience in proving or showing that a numerical statement or process is correct. The questions also express in words a mathematical process with which pupils have been working. Drawings and multiple ways of finding the answer are stressed. It may be further noted that the questions encourage investigation and exploration for a solution. The emphasis is upon having pupils make suggestions, ask questions, test propositions, and challenge others' views. Providing pupils with the idea that a variety of methods may be used to get the answers to the exercises is believed to have educational value. The ability to use more than one method is presumably an indication of a good understanding of numbers, processes, and problem situations. The experience of selecting and using a method which seems best for the particular arithmetical situation encourages clear reasoning and gives pupils a chance to exercise good judgment. Various methods enable pupils to generalize and formulate explanatory statements which are meaningful descriptions of the process.

Other characteristics of the review questions are found upon perusal. These include the use of several ways to check the correctness of solutions; encouragement of perhaps cumbersome but known ways to find the answers to questions; identification of various specific ways to study; use of such special aids as the number line; and the like.

Suggestions for procedure with the review study questions are offered teachers who are interested:

1. Present the review questions without elaborate explanations or directions. The following teacher directions represent a desirable procedure:

"Today we are to use our arithmetic period to answer the questions on this study sheet. These are questions about the arithmetic we have just been studying."

After the papers have been passed out and a few minutes have elapsed for pupils to try some of the exercises, the teacher might say, "Let's see if we understand the assignment" or "Let's see how we are doing."

This will give the teacher an opportunity to check on pupil understanding of the assignment. The teacher may discuss the first or second question with the pupils to determine this understanding. Satisfied that they do, the teacher may indicate that the study period is to proceed.

2. Observe the individual pupils during the study period and work with any who indicate need for help. Call the pupil's attention to what he might do if he could not answer or was uncertain of the answer to a question.

3. Ask individual pupils (possibly the faster workers) to place on the board during the work period solutions for use in the discussion period. These could be answers to a "showing" or "proving" question.

4. Begin the discussion period with a statement such as "Now that the work period is ended, let's check and discuss the answers. Let's see the ways of showing that an answer is correct." Pupils will be called upon to give the answers to the questions. The teacher will often be saying to these answers, "How did you think to get to your answer?", "How many ways of showing it can you think of?", "Did anyone use a different way to find the answer?", "Put your way on the chalkboard for all of us to see," "Do you think Billy used a good method to find the answer?" and the like.

5. Make an effort to include all the pupils in the discussion.

6. Try to avoid accepting partial, poorly worded statements as answers to questions. The use of such procedures as asking other pupils to restate what has been said is a good way to point toward the kind of clear thinking desired.

Teachers who have used this approach to review feel that such questions help motivate and enrich the classroom work. They believe that review study questions stimulate interest; improve accuracy, clearness, and organization of meanings and concepts; permit a more active and aggressive attitude toward learning; challenge critical thought; and afford a basis by which pupils and teachers may appraise results. Pupils appear to enjoy the questions because they are a change in procedure.

In summary, the purposes and general values of review study questions are many.

a. Review study questions may serve as a means of initial learning for those who have had difficulty in developing a concept or in becoming skillful in using an idea.

b. Review study questions may serve as a stimulating summarizing device, helping to identify and clarify points not yet understood, and extending understanding of past learnings.

c. Review study questions may serve to provide recall as an aid in remembering.

d. Review study questions may provide a means of checking a pupil's understanding of the arithmetic that has been studied.

e. Review study questions may serve as a basis for valuable discussion about various aspects of number work.

f. Review study questions may encourage investigation and exploration for a solution, utilizing the scientific approach to problem solving.

It seems appropriate, therefore, to suggest that teachers consider review study questions as one good way to encourage pupils to think about the materials they have studied and organize their learnings.

EDITOR'S NOTE: Certainly all of us would subscribe to the practice of using interesting and varied questions to make the learner examine his beliefs and justify his answers. The ingenuity and effort necessary to concoct such questions is often lacking in us. These models ought to help many teachers develop such a practice.

Some readers will probably think they can improve upon these very questions. That might be a good way to begin. It is surprising how our own shallow understandings show through when we try to phrase questions that really get to the bottom of things.

Enrichment in Arithmetic for the Primary Grades

(*Concluded from page 413*)

That some change is needed appears apparent. All children should no longer be limited to a course of study developed to meet the ability of slower or even average children. We must in some way cease to limit bright children to "horizontal enrichment." There just isn't enough content in the lower grades at the present time to enrich only horizontally. Too many children are being held back by this type of restriction.

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EDITOR'S NOTE: Another article points to the inadequate diet reputedly offered young children in arithmetic. There is no doubt in the light of the figures quoted here, or in the light of the experience many of us have with youngsters in school and out, that these charges are serious.

The answer to the question of what to do for all the levels we must provide for may involve a considerable disturbance of the present school organization and routine. Ungraded primary is one proposal.

Articles such as W. L. Whitaker's (also in this issue) present ideas worth trying where such reorganization is not possible.

Selected Educational and Experience Factors and Arithmetic Teaching

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WHILE THE PROFOUND CHANGES NOW being made in the teaching of secondary school mathematics will soon have counterparts in the teaching of elementary school mathematics, the effectiveness of the elementary mathematics program is and will remain contingent on the mathematics background and teaching skill of the elementary classroom teacher.

Purpose of the Study

This study attempted to discover the relationship of selected teacher background factors as determinants of successful elementary school arithmetic instruction:

1. Is the number and type of high school mathematics courses taken a factor in successful elementary school arithmetic teaching?
2. Is the number of college mathematics courses taken (exclusive of arithmetic methods courses) a factor?
3. Is the number of years of elementary teaching experience a factor?
4. Is the teacher's attitude toward high school mathematics courses taken a factor?
5. Is the teacher's self-image a factor?

Related Research

Research on the background of teachers of arithmetic is limited in amount. No research which relates directly to our topic has been done. The lack of formal preparation in mathematics by most teachers was revealed by Layton [4]* and Grossnickle [3]. Studies by Dutton [2] and Orleans [5] indicated lack of confidence and unfavorable attitudes toward arithmetic on the part of those who teach it.

Standlee and Popham [6], correlating teaching success and some attributes of background, found that no significant relationship existed between grade point average of academic or professional courses and principal's rating of teacher performance. Steinbrook [7] in an unpublished Doctoral Dissertation (Indiana, 1955) concluded that teaching experience appeared to be associated with success in teaching and that teaching success seemed to depend upon a number of complex factors, of which none was totally differentiating. Bean [1] tested the understanding level in mathematics and found there was a small cumulative increase in teachers' understanding of arithmetic, reaching a peak with eleven to fifteen years of experience.

Methods and Procedure

In personal interview with the investigators, 66 elementary school principals in ten school districts in San Mateo and Santa Clara counties were asked to evaluate the arithmetic teaching competency of their fourth-grade teachers as "superior," "good," or "fair."

A questionnaire was sent to the 157 fourth-grade teachers to ascertain: (1) number of years of elementary teaching experience, (2) number and type of high school mathematics courses taken, (3) number of college mathematics courses taken (exclusive of arithmetic method courses), (4) attitude toward high school mathematics courses taken, (5) self-rating as a "superior," "good," or "fair" arithmetic teacher.

Of the 157 teachers reached, 102, or 65%, returned the questionnaire. They comprise our sample.

* Figures in brackets refer to items in the bibliography at the end of this article, p. 430.

TABLE I

COMPARISON OF PRINCIPALS' EVALUATIONS AND TEACHERS' SELF-RATINGS

Principals' Evaluations	Self-Images by 102 Teachers			
	Superior	Good	Fair	No Answer
Superior	16.2%	73.0%	5.4%	5.4%
Good	17.3	71.2	11.5	0.0
Fair	0.0	53.8	30.8	15.4

Findings

Interrogation of the building principals disclosed the following percentage ratings in each of the three established classifications for the 102 teachers: *superior*, 36.3%; *good*, 51.0%; *fair*, 12.7%.

Comparison of the principals' evaluations and the teachers' self-ratings is shown in Table I.

Examination of the data indicated that teachers judged superior by the principal tended to underrate their ability and teachers judged fair by the principal tended to overrate their ability.

The number and type of high school mathematics courses taken by teachers evaluated by the principals in each of the three established classifications are found in Table II.

More fair arithmetic teachers (69.2%) than good teachers (63.5%) or superior teachers (51.4%) completed general mathematics, a course traditionally taken by less able students. The over-all data revealed no significant differences between the amount and type of high school mathematics taken and arithmetic teaching ability.

Comparison of teacher attitude toward high school mathematics courses taken and principal's evaluation of teaching competency is shown in Table III.

Fair teachers seem to have a more negative attitude toward high school mathematics experiences than good or superior teachers. Superior teachers seem to have a more positive attitude toward high school mathematics experience than good or fair teachers.

TABLE II

HIGH SCHOOL MATHEMATICS COURSES TAKEN BY 102 TEACHERS

Principals' Evaluations	General Math	Algebra I	Geometry	Algebra II	Solid Geometry	Trigonometry
Superior	51.4%	100.0%	91.9%	48.6%	18.9%	16.2%
Good	63.5	94.2	92.3	48.1	26.9	15.4
Fair	69.2	100.0	84.6	46.2	15.4	23.1

TABLE III

TEACHERS' ATTITUDES TOWARD HIGH SCHOOL MATHEMATICS

Principals' Evaluations	Extremely Interested	Moderately Interested	Disinterested	Mixed Feelings
Superior	48.7%	43.2%	8.1%	0.0%
Good	34.6	46.2	9.6	9.6
Fair	7.7	69.2	23.1	0.0

TABLE IV

COMPARISON OF TEACHING EXPERIENCE AND QUALITY OF INSTRUCTION

Principals' Evaluations	Years of Experience for 102 Teachers								
	1–2	3–4	5–6	7–8	9–10	11–12	13–14	15–16	17–30
Superior	13.7%	13.7%	13.7%	8.0%	16.0%	0.0%	8.0%	5.4%	21.6%
Good	36.5	21.2	7.7	3.9	9.6	7.7	1.9	5.8	5.8
Fair	30.8	53.8	7.7	0.0	7.7	0.0	0.0	0.0	0.0

Table IV shows a comparison of number of years of elementary teaching experience and the quality of instruction.

The median number years of teaching experience of superior teachers was nine, of good teachers four, and fair teachers three. The data indicated that years of experience improved the quality of arithmetic instruction.

Number of college mathematics courses (exclusive of arithmetic method courses) taken by teachers in comparison with the principals' evaluations in each of the three established classifications is the subject of Table V.

The data indicated no significant differences between the number of college mathematics courses taken and arithmetic teaching ability.

Summary and Conclusions

All of the following conclusions are based on the premise that principals accurately appraised and differentiated the level of the teachers' arithmetic instructional performances:

1. Most of the interviewed principals felt that their fourth-grade arithmetic programs were adequate as indicated by their rating 87.3% of their teachers "superior" or "good." However, since 63.7% of the fourth-grade teachers were rated less than "superior," and since many of the superior teachers could probably be even better, room for improvement still exists.

2. Superior teachers tend to underrate themselves and fair teachers to overrate themselves.

3. No significant relationship was demonstrated between number and type of high school mathematics courses completed and the quality of the teachers' arithmetic instructional programs at the fourth-grade level. Superior and good teachers were somewhat less apt to have taken high school general mathematics than were fair teachers.

4. No significant relationship was demonstrated between the number of college mathematics courses taken and the quality of the teachers' arithmetic instructional programs at the fourth-grade level.

5. The superior arithmetic teachers demonstrated an appreciably higher degree of interest in their high school mathematics courses than either the good or the fair teachers. A negative attitude was much more prevalent among fair teachers.

(*Concluded on page* 430)

TABLE V

COLLEGE CONTENT MATHEMATICS COURSES COMPLETED BY 102 TEACHERS

Principals' Evaluations	No Units	1–6 Units	7–12 Units	More Than 12
Superior	70.3%	24.3%	5.4%	0.0%
Good	63.5	23.1	7.6	5.7
Fair	76.9	23.1	0.0	0.0

The Training of Elementary-School Mathematics Teachers

An Abridgment of the Recommendations of the Mathematical Association of America for the Training of Teachers of Mathematics

THE COMMITTEE on the Undergraduate Program in Mathematics (CUPM)[1] is a committee of the Mathematical Association of America and is supported in part by the National Science Foundation. The general purpose of this committee is to develop a broad program of improvement in the undergraduate mathematics curriculum of the nation's colleges and universities.

As part of its mandate, CUPM established a panel on teacher training.[2] This panel was instructed to prepare for CUPM a set of recommendations of minimum standards for the training of teachers on all levels. The following report, dated August 1960, is the result of the work of the panel on teacher training and has received the endorsement of the Committee on the Undergraduate Program in Mathematics and of the Board of Governors of the Mathematical Association of America.

The panel on teacher training has been further charged with the implementation of these recommendations and hopes to issue supplementary reports, as well as to hold various regional conferences, to make these minimum standards a reality.

The report consists of the following parts: general recommendations, the Five Levels, recommendations for the Five Levels, summary of recommendations, curriculum-study course, training of supervisors, sample course descriptions.

Reprints and further information may be obtained by writing the Executive Director: Professor Robert J. Wisner, Michigan State University Oakland, Rochester, Michigan.

General Recommendations

The purpose of this report is to present a preliminary outline of the panel's recommendations for the minimal college training program for teachers of mathematics. We have found it a most useful device to classify mathematics teachers according to their position in an over-all sequential schedule of presenting the main ideas of mathematics.

For each classification, we give a recommendation as to the type and minimum amount of mathematics which should be taken by the student preparing for a teaching career. The courses described are specifically designed for prospective teachers and should be taught by persons who are masters of their subject matter and who have, in addition, a knowledge of the problems which teachers face.

These sample courses are given solely for illustrative purposes to explain the type of courses and the levels of advancement desirable for prospective teachers. It should be clearly understood that different institutions will wish to exercise considerable freedom in implementing these recommendations,

[1] Members are: R. C. Buck, chairman, and R. J. Wisner, executive director; E. G. Begle, L. W. Cohen, W. T. Guy, Jr., R. D. James, J. L. Kelley, J. G. Kemeny, E. E. Moise, J. C. Moore, Frederick Mosteller, H. O. Pollak, G. B. Price, Patrick Suppes, Henry Van Engen, R. J. Walker, and A. D. Wallace.

[2] Members of the teacher training panel include: John G. Kemeny, chairman, Dartmouth College; E. G. Begle, Yale University; W. T. Guy, Jr., University of Texas; P. S. Jones, University of Michigan; J. L. Kelley, University of California, Berkeley; Bruce Meserve, Montclair State College; E. E. Moise, Harvard University; Rothwell Stephens, Knox College; Henry Van Engen, University of Wisconsin.

Ex officio members of the panel are: R. Creighton Buck, University of Wisconsin, chairman; and Robert J. Wisner, Michigan State University Oakland, executive director.

both as to the way topics are combined into courses and as to the exact choice of topics for individual courses.

There are several sincere warnings put forth in regard to the reading and interpretation of this report.

First, the classifications are to be taken in the rather loose fashion in which they are described. Their exact delineations will, of course, depend upon local conditions of school and curricular organization. It should be noted that the various classifications overlap: this is done deliberately in an attempt to meet just such local conditions.

Second, the recommendations are not motivated by a desire to meet the demands of any special program of mathematics education; nor do the descriptions or outlines of courses to be taken by prospective teachers represent an attempt on the part of this committee to further goals of any particular school curriculum planning organization. The recommendations are meant to be the minimum which should be required of teachers in any reasonable educational program, and the course descriptions are presented only to illustrate what is meant by the course titles.

Third, it is to be hoped that everyone recognizes good mathematics education to be a sequential experience. Thus, the teacher at any particular level should have an understanding of the mathematics which will confront the students in subsequent courses; and as a consequence, it is desirable that a teacher at a given level be prepared to teach at least some succeeding courses. Ideally, a person preparing for teaching should meet in addition to the minimal requirements set forth here as many of the requirements for the next level as his or her college program permits.

Fourth, this report is meant as a guide to the preparation of people who will be teaching any mathematics whatsoever. The suggestions apply, within each level, to all people who teach any mathematics. The recommendations do not in any sense exclude the teacher who is assigned classes scheduled primarily for students of low aptitude.

Fifth, the phrase "a course" occurs in several places in this report. For purposes of fixing ideas, this phrase is employed in the sense of a three semester-hour presentation of the subject matter described, and it is not meant to exclude integrated programs or other curricular arrangements.

Finally, *the reader should note that the training for Level I teaching is a separate program, while the curricula for the further levels form a cumulative sequence, in which each program is a continuation of the preceding one.* [Italics added by the editor.]

The committee benefited greatly from such previous studies on teacher preparation as that of the Co-operative Committee on the Teaching of Science and Mathematics, a committee of the American Association for the Advancement of Science. The committee was also guided by discussions with a variety of professional organizations. It is pleased to note the considerable degree of agreement common to all proposals.

It should be emphasized that these recommendations are minimal in nature and that some institutions have already met and exceeded these recommendations. It is expected that as high school curricula are strengthened, these minimum recommendations will be revised.

The Five Levels

I. Teachers of elementary school mathematics

This level consists of teachers confronted with the problem of presenting the elements of arithmetic and the associated material now commonly taught in grades K through 6. The committee recognizes that special pedagogical problems may be connected with grades K through 2 and so a special program may be appropriate for teachers of such grades.

II. Teachers of the elements of algebra and geometry

Included here are teachers who are assigned the task of giving introductory-year courses in either algebra or geometry or the less formal preliminary material in these fields. These introductory courses are now commonly taught in grades 7 through 10.

III. Teachers of high school mathematics

These teachers are qualified to teach a modern high school mathematics sequence[3] in grades 9 through 12.

IV. Teachers of the elements of calculus, linear algebra, probability, etc.

This is a mixed level, consisting of teachers of advanced programs in high school, junior college teachers, and staff members employed by universities to teach in the first two years. These teachers should be qualified to present a modern two-year college mathematics program.

V. Teachers of college mathematics

These teachers should be qualified to teach all basic courses offered in a strong undergraduate college curriculum.

. . . . . .

The levels having been presented, we are now ready to proceed to a description of our recommendations of the minimal college training requirements for entry into the teaching profession at each level.

Recommendations for Level I

(Teachers of Elementary School Mathematics)

As a prerequisite for the college training of elementary-school teachers, we recommend at least two years of college preparatory mathematics, consisting of a year of algebra and a year of geometry, or the same material in integrated courses. It must also be assured that these teachers are competent in the basic techniques of arithmetic. The exact length of the training program will depend on the strength of their preparation. For their college training, we recommend the equivalent of the following:

(a) A course or a two-course sequence devoted to the structure of the real number system and its subsystems (Course-Sequence 1).[4]
(b) A course devoted to the basic concepts of algebra (Course 2).
(c) A course in informal geometry (Course 3).

The material in these courses might, in a sense, duplicate material studied in high school by the prospective teacher, but we urge that this material be covered again, this time from a more sophisticated, college-level point of view.

Whether the material suggested in (a) above can be covered in one or two courses will clearly depend upon the previous preparation of the student.

We strongly recommend that at least 20 per cent of the Level I teachers in each school have stronger preparation in mathematics, comparable to Level II preparation, but not necessarily including calculus. Such teachers would strengthen the elementary program by their very presence within the school faculty. This additional preparation is certainly required for elementary teachers who are called upon to teach an introduction to algebra or geometry.

Recommendations for Level II

(Teachers of the Elements of Algebra and Geometry)

Prospective teachers should enter this program ready for a mathematics course at the level of a beginning course in analytic geometry and calculus (requiring a minimum

[3] Such sequences have been recommended by the Commission on Mathematics, the School Mathematics Study Group, the University of Illinois Committee on School Mathematics, and others.

[4] Sample courses, by numbers, are to be found in the Appendix. More extensive descriptions of them are being prepared for later publication.

of three years in college preparatory mathematics). It is recognized that many students will have to correct high school deficiencies in college. (However, such courses as trigonometry and college algebra should not count toward the fulfillment of minimum requirements at the college level.) Their college mathematics training should then include:

(a) Three courses in elementary analysis (including or presupposing the fundamentals of analytic geometry). (See Course-Sequence 4.) This introduction to analysis should stress basic concepts. However, prospective teachers should be qualified to take more advanced mathematics courses requiring a year of the calculus and hence calculus courses especially designed for teachers are normally not desirable.

(b) Four other courses: a course in abstract algebra, a course in geometry, a course in probability from a set-theoretic point of view, and one elective. One of these courses should contain an introduction to the language of logic and sets. (See Courses 5–7.)

. . . . . .

[EDITORIAL NOTE: *Descriptions of the recommendations and courses for Levels III, IV, and V are given in the December, 1960 issue of* The Mathematics Teacher.]

Curriculum-Study Courses

The above recommendations have dealt in detail with the subject-matter training of mathematics teachers. There are many other facets to the education of the scholarly, vigorous, and enthusiastic persons to whom we wish to entrust the education of our youth. One of these merits special mention by us. Effective mathematics teachers must be familiar with such items as:

(a) The objectives and content of the many proposals for change in our curriculum and texts.

(b) The techniques, relative merits, and roles of such teaching procedures as the inductive and deductive approaches to new ideas.

(c) The literature of mathematics and its teaching.

(d) The underlying ideas of elementary mathematics and the manner in which they may provide a rational basis for teaching, unless these are taken care of by mathematics courses especially designed for teachers.

Such topics are properly taught in so-called "methods" courses. We would like to stress that adequate teaching of these can be done only by persons who are well informed *both* as to the basic mathematical concepts and as to the nature of American public schools, as well as to the concepts, problems, and literature of mathematics education. In particular, we do not feel that this can be done effectively at either the elementary or secondary level in the context of "general" methods courses, or by persons who have not had at least the training of Level IV.

Training of Supervisors

There is a great need for providing adequately trained supervisors of mathematics, grades K through 12, for our public schools. At present, administrators find no ready supply of such individuals and, hence, are through necessity making appointments which are highly questionable, if not indefensible. For this reason, it is urgent to develop a program for supervisors and to seek adequate support for those individuals who have the desired qualities for supervision and the ability to benefit from advanced training. Such training would prepare the "leaders of teachers" in the local system, (a) to make sound judgments concerning mathematics programs for the schools, (b) to understand thoroughly the recommendations made by national committees, and (c) to enable schools better to articulate school mathematics with college mathematics.

Prerequisite to this program should be a regular Master's degree in mathematics or

a Master's degree given as a result of participation in an Academic Year Institute. The program should consist of additional graduate courses in abstract algebra, analysis, and geometry, with courses selected from logic, statistics, theory of numbers, philosophy of education, history of education, history of mathematics, seminar courses on the program of the elementary school and secondary school mathematics, and additional elective courses in algebra, analysis, or geometry to provide some degree of concentration.

The committee feels that action must be taken to fill the need for supervisory personnel, and we recommend such action to the appropriate authorities.

Appendix: Course Descriptions

We list below sample courses that might be used to fulfill the minimum requirements for Levels I through III, and the undergraduate requirements of Levels IV and V. These brief descriptions are included to clarify the meaning of course titles but are not intended as syllabi for actual courses. It must be recognized that there are other equally good ways of combining various recommended topics, and colleges should be encouraged to work out detailed curricula to suit their own tastes and local conditions. However, the committee hopes that these very brief descriptions will help in indicating the types of courses desirable and the level of advancement.

For Level I

1. *Algebraic structure of the number system* (2-course sequence). This is a study of the numbers used in elementary school—whole numbers, common fractions, decimal fractions, irrational numbers.

Emphasis should be on the basic concepts and techniques: properties of addition, multiplication, inverses, systems of numeration, and the number line. The techniques for computation with numbers should be derived from the properties and structure of the number system, and some attention should be paid to approximation. Some elementary number theory, including prime numbers, properties of even and odd numbers, and some arithmetic with congruences should be included.

2. *Algebra.* Basic ideas and structure of algebra, including equations, inequalities, positive and negative numbers, absolute value, graphing of truth sets of equations and inequalities, examples of other algebraic systems—definitely including finite ones—to emphasize the structure of algebra as well as simple concepts and language of sets.

3. *Intuitive foundations of geometry.* A study of space, plane, and line as sets of points, considering separation properties and simple closed curves; the triangle, rectangle, circle, sphere, and other figures in the plane and space considered as sets of points with their properties developed intuitively; the concept of deduction and the beginning of deductive theory based on the properties that have been identified in the intuitive development; concepts of measurement in the plane and space, angle measurement, measurement of the circle, volumes of familiar solids; treatment of co-ordinate geometry through graphs of simple equations.

. . . . . .

Attend the Annual Christmas Meeting

Plan to attend the Nineteenth Christmas Meeting of The National Council of Teachers of Mathematics to be held at Arizona State College, Tempe, Arizona, December 27–30, 1960.

In the Classroom

Edited by EDWINA DEANS
Arlington Public Schools, Arlington, Virginia

Providing Meaningful Practice

AS A CLASSROOM TEACHER, you are continuously faced with the problem of providing meaningful practice which will help children maintain facts learned in the four fundamental processes. Practice consisting mainly of repetition and rote learning becomes deadly and uninteresting for many children. When children are presented with exercises which require only repetition and recall, they tend to lose their intellectual curiosity about arithmetic and perform as automatons rather than as thinking and reasoning individuals.

The suggestions for exercises presented in this issue are an attempt to use the framework of the basic facts to help children understand mathematical principles. As children work through similar exercises, they will get the practice needed to deepen understanding and maintain facts. They will also learn principles applicable to odd and even numbers and to equal and unequal quantities. You will be aware of applications of the commutative, associative, and distributive laws of mathematics to the practice materials.

Some of the suggestions presented in this issue are adapted from reports prepared by participants in the Arlington County Arithmetic Workshop held during the summer of 1960. Contributions by the following teachers are gratefully acknowledged: Orleta Coston, Alice Mercer, and Dorothy Pully.

A. Provide practice circles that focus attention on adding and subtracting odd and even numbers.

1. Draw the practice wheel, such as that in Figure 1, on the chalkboard or on a chart. Children add the center number to each of the numbers in the second circle. Answers go in the third circle.

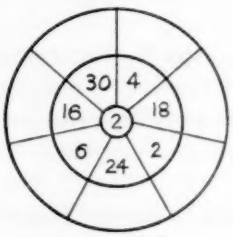

FIG. 1

Ask questions to help children concentrate on the kinds of numbers involved:

What do you notice about each number in the second circle?

What kind of numbers are they? (All are even numbers.)

Is 2 the same kind of number? (Yes.)

What happens when you add 2 to each number in the second circle? (You get the next even number. The answer is 2 bigger. The answer is the same kind of number.)

Change the numbers in the second circle and repeat. Ask the questions again.

2. Have children suggest odd numbers to place in each space around the second circle, as in Figure 2. (1, 3, 5, 7, 9, 11, 23, 25, 37, 39, 41, etc., may be suggested.)

Again ask questions about the kind of numbers:

What kind of numbers are in the second circle? (Odd numbers.)

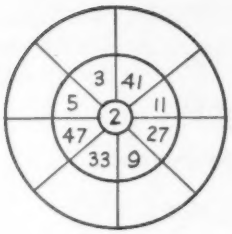

FIG. 2

Is 2 the same kind of number? (No, 2 is an even number.)

What kind of numbers are the answers? (All are odd numbers.)

What else do you notice about the answers? (When 2 is added to an odd number, you get the next odd number. The answers are 2 bigger. All the answers are odd numbers.)

3. Children may build their own practice circles and describe the kind of numbers used, or they may build circles according to instructions you provide. The following are samples of instructions that can be used:

   a. The center number must be an odd number between 3 and 9. The numbers in the second circle must all be even numbers between 24 and 60. Find the answers. Study the answers to see what kind of numbers you have. (All will be odd numbers.)

   b. The center number must be an odd number greater than 6, but less than 10. The numbers in the second circle must be odd numbers greater than 30 and less than 50. Find the answers. Study the answers to see what kind of numbers you have. (All will be even numbers.)

   c. Shorten instructions by using symbols $>$ (greater than) and $<$ (less than). Center number: 6. Circle numbers: Even numbers $>$ 10 and $<$ 30.

4. Subtraction, multiplication, and division circles may be prepared in the same way to give practice on the following:

   a. Even number minus even number.
   b. Odd number minus even number.
   c. Odd number minus odd number.
   d. Even number times even number.
   e. Even number times odd number.
   f. Odd number times odd number.
   g. Even number divided by even number.
   h. Even number divided by odd number.
   i. Odd number divided by odd number.

B. Have children develop sets of facts that meet definite requirements.

1. $n+4>14$

   What set of numbers plus 4 is greater than 14?

   Replace n with at least 6 different numbers.

2. $n+4<14$

   What set of numbers plus 4 is less than 14?

   Replace n with all the numbers that will work.

3. $\triangle+\square=10$

   Replace $\triangle$ and $\square$ with numbers.

$\triangle$	$\square$
1	9

   Rule: If the shape is different, the numbers must be different.

   If $\triangle$ is 1, what is $\square$?(9)

   If $\triangle$ is 2, what is $\square$?(8)

   Repeat for any sets of facts desired:

   $\triangle+\square=15$; $\triangle+\square=13$

C. Use equations to emphasize the commutative property of numbers.

1. Instruct children to select numbers for the boxes to make the number sentence true. Have them find answers

for both sides of the equation to prove the sentence is true. If more than one solution is possible, designate the number desired.

Illustration:

$3+7=7+\square$

$3+7=7+3$

$10=10$

Sample exercises:

$\square+9=9+\square$

(Solve with 3 different numbers)

$6\times\square=8\times6$

$24=6\times\square=\square\times6$

$3\times\square=\square\times3$

$2\times\square=\square\times2$

D. Give practice in finding equal quantities.

1. Have children solve each side of the equation to prove their answers.

Illustration:

$8+6=9+\square$

$8+6=9+5$

$14=14$

Sample exercises:

$10-\square=2\times\square$

$15-\square=7+\square$

$3\times\square=\square+6$

$3+(6\times2)=3+(\square\times6)$

2. Have children select the sign to show whether two quantities are equal or unequal.

Select the sign = (equal) or $\neq$ (unequal) for these number sentences. Prove each by finding answers for both sides of the equation.

Illustrations:

$8\times3\square6\times4$

$8\times3=6\times4$

$24=24$

$5\times9\square10\times4$

$5\times9\neq10\times4$

$45\neq40$

Sample exercises:

$9\times8\square12\times6$

$\frac{36}{9}\ \square\ \frac{18}{3}$

$\frac{35}{7}\ \square\ \frac{45}{9}$

E. Give practice exercises which children can solve by one method and prove by another.

1. Add the columns from the top down. Draw a line through each number that completes a 10. Add the excess of 10 to the next number.

Solution:

$$\begin{array}{r} 7 \\ \not{5} \\ 3 \\ \not{8} \\ \not{9} \\ \hline 32 \end{array}$$

$7+5=12$ or $10+2$; put a line through 5 to show 10; $2+3+8=13$; put a mark through 8 to show another 10; $3+9=12$; put a line through 9 to show another 10; record 2 in one's place; count the marks to get the number of tens.

2. Check by adding the column from bottom to top in the regular way.

F. Provide puzzle type practice exercises for variety.

1. Triangle puzzle, shown in Figure 3, page 429.

Your sum for each side should be 13. Place a number at each dot. Use

a number only once. Select a number so that the sum of each side of the triangle will be 13.

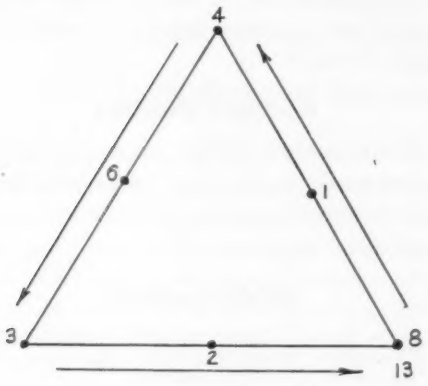

FIG. 3

2. Missing number puzzles.

Rule: Use the same number in the same shape.

a.

8	+	7	=	□
□	−	6	=	△
△	+	3	=	○
△	+	○	=	□

Solution:

$8+7=15$; $15-6=9$;

$9+3=12$; $9+12=21$

b.

16	−	7	=	□
□	+	5	=	△
△	−	8	=	○
△	+	○	=	20

Solution:

$16-7=9$

$9+5=14$

$14-8=6$

$14+6=20$

c.

3	×	8	=	□
□/4			=	△
△	×	8	=	○
○/△			=	8

Solution:

$3\times8=24$

$\frac{24}{4}=6$

$6\times8=48$

$\frac{48}{6}=8$

G. These exercises, emphasizing the associative principle, help children regroup one addend to complete a ten.

$8+6=(8+2)+4 \quad 8+2=10+4=14$

$7+9=(7+3)+6 \quad 7+3=10+6=16$

H. Use exercises stressing the distributive principle to help children learn multiplication facts which are difficult for them.

1. $\begin{array}{r} 8 \\ \times 7 \\ \hline (3+4) \end{array} = \begin{array}{l} (2+5) \\ \\ \end{array} \begin{array}{l} (2\times8)+(5\times8) \\ 16+40=56 \\ = (3\times8)+(4\times8) \\ 24+32=56 \end{array}$

$\begin{array}{r} 9 \\ \times 8 \\ \hline (5+3) \end{array} = \begin{array}{l} (4+4) \\ \\ \end{array} \begin{array}{l} (4\times9)+(4\times9) \\ 36+36=72 \\ 36+36 \\ = (5\times9)+(3\times9) \\ 45+27=72 \end{array}$

2. $6\overline{)48}$

30	5	(5 six's = 30)
18		
18	3	(3 six's = 18)
0		
	8	(8 six's = 48)

$8\overline{)72}$

40	5	(5 eight's = 40)
32		
32	4	(4 eight's = 32)
0		
	9	(9 eight's = 72)

The suggestions offered can be adapted for several grade levels. The content for each exercise can be selected to reinforce the abilities being developed at each grade level.

Selected Educational and Experience Factors and Arithmetic Teaching

(*Concluded from page 420*)

6. Years of teaching experience improve the quality of arithmetic instruction. The median number of years of experience of superior teachers was nine, of good teachers four, and of fair teachers three. Seventy-four per cent of the teachers with more than 17 years' experience were judged superior. No teacher with this amount of experience was evaluated as fair.

Our conclusions suggest these implications:

1. In view of the facts that San Mateo and Santa Clara counties are considered desirable places to teach and qualifications for employment are probably somewhat higher than average, room for improvement would probably be even higher in arithmetic teaching at this level in some other geographical areas.

2. How much of the underrating of superior teachers was due to modesty and overrating by the fair teachers due to wishful thinking was not revealed in this study. The self-image on the part of the fair teacher that shows him doing an adequate job indicates complacency which will be a hindrance to professional growth and in-service training.

3. In pre-employment interviews, an effort to obtain the teacher's attitude toward high school mathematics experiences would seem advisable.

Further Studies

Future study is needed at the seventh- and eighth-grade levels, since new variables of organizational patterns for instruction are introduced at these levels.

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EDITOR'S NOTE: It is interesting to speculate upon what makes good teachers.

It may be surprising that the courses in mathematics which the teachers in this study had taken show so little relation to their rated abilities.

The only real anomaly in the figures, that having to do with the trigonometry taken by fair teachers, was explained by the authors as probably being a chance deviation, unimportant because of the small number of cases at this place in the table.

The authors' confidence in the sixth conclusion exceeds ours. However, they very properly defend against the argument that this increase may be by attrition of poor teachers, who drop out quickly, by pointing out that many superior teachers also leave the ranks of classroom teachers for administrative or supervisory positions.

Maybe they are right. After all, it would be too bad if we all got worse and worse as the years go by.

Research and Curriculum Improvement

Edited by J. Fred Weaver
Boston University School of Education, Boston, Massachusetts

A Non-Grade-Level Sequence in Elementary Mathematics

The December, 1959, issue of The Arithmetic Teacher carried an account of "The Dual Progress Plan in the Elementary School,"* one feature of which is a nongraded sequence for mathematics instruction. This specific feature is described in greater detail in a more recent report (July 15, 1960) on "A Non-Grade-Level Sequence in Elementary Mathematics," prepared by Glen Heathers, Director of the Experimental Teaching Center at New York University, and Samuel Steinberg, Mathematics Consultant to the Center.

The first part of the Heathers-Steinberg report presents background information on the development of a nongraded sequence in elementary mathematics which, in its present preliminary form, is arranged in a series of seventeen major levels (topics), each of which is divided into numerous sublevels (subtopics). These major levels and their sublevels reportedly have been arranged in a sequential learning order on the basis of three principal criteria: "(1) the internal logic of mathematics, (2) the difficulty of the material, and (3) the maturity of the student."

The following seventeen major levels of the tentative non-grade-level sequence in elementary mathematics are given.

1. Meaning and use of numbers from one to ten
2. Meaning and use of numbers from eleven to one hundred
3. Addition with carrying; subtraction with exchanging
4. Multiplication and division by one-place multipliers and divisors
5. Meaning and use of common fractions
6. Geometry, graphs, and maps
7. Meaning and use of large numbers
8. Multiplication by two-place multipliers; division by two-place divisors
9. Comparing fractions
10. Operations with unlike fractions
11. Meaning and use of decimal fractions
12. Graphs, maps, and Cartesian coordinates
13. Multiplication and division by fractions
14. Ratios and percentages
15. Squares and square roots
16. Variables and number sentences
17. Probability and statistics

The second part of the report is devoted to a partially annotated breakdown of each of these major levels into various sublevels. It is planned to develop more complete analyses and annotations, and to extend the levels to cover a nongraded sequence in mathematics for grades K through 12.

Interested persons may secure a copy of the present two-part report on "A Non-Grade-Level Sequence in Elementary Mathematics" by writing to Dr. Glen Heathers, Director, Experimental Teaching Center, New York University, School of Education, Washington Square, New York 3, New York. Also, as the report states: "The Experimental Teaching Center invites interested persons to participate in revising and extending the sequence, or in annotating it in any way that will make it more useful to the classroom teacher. The staff of the Center desires to work with others who wish to share in developing a non-grade-level mathematics sequence for elementary and secondary schools."

* Glen Heathers and Morris Pincus, "The Dual Progress Plan in the Elementary School," The Arithmetic Teacher, VI (December, 1959), 302–5.

The National Council of Teachers of Mathematics

Membership Report

The following is a portion of the Annual Report submitted to the members of The National Council of Teachers of Mathematics at the annual meeting in Buffalo, April, 1960, by Mary C. Rogers, Chairman, Membership Committee.

The final membership count for the year —as of May 1, 1960—shows a total of 31,194 members and subscribers, or 3,194 more than the larger goal of 28,000 which we had set for ourselves. Currently, individual memberships comprise about 72% of grand totals. You will be interested to know that we are now the second largest department affiliated with the NEA. A year from now we should be the largest.

Examining the year-by-year growth in grand totals during the past six years, we find:

Increase from May 1954 to May 1955	2,723
Increase from May 1955 to May 1956	1,103
Increase from May 1956 to May 1957	2,354
Increase from May 1957 to May 1958	3,316
Increase from May 1958 to May 1959	4,293
Increase from May 1959 to May 1960	7,404

Action has been taken to accept the recommendation for a 40,000 membership goal. A time limit of one year is set for the accomplishment of this new objective.

Making a detailed analysis of National Council membership achievement by state and territory, one notes that:

41 states and territories are well over the current 25,000 goals;

8 states and territories need fewer than 25 members to reach their goals;

5 states have not been so successful and need the help of individuals in those states;

24 states have had continuous growth during the past years;

10 states are already well over their 40,000 goals.

We have 3,160 student members. We are pleased about this, for we are convinced that this interest will continue with these young people as our future teachers of America.

The following tables give a more detailed picture of NCTM membership. You may study them for the specific details which are of interest to you.

The Membership Committee again extends to you its sincere thanks for your outstanding co-operation and helpfulness.

Pearl Bond	Lucille Houston
Marian C. Cliffe	Florence Ingham
Mary Lee Foster	Mildred Keiffer
Janet Height	Faith Novinger
Harold J. Hunt	Bess Patton

M. H. Ahrendt, Ex Officio member
Eugene P. Smith, Ex Officio member
Mary C. Rogers, *Chairman*

Leaders in Membership Totals
(Including Subscriptions)

New York	2,893	Texas	1,425	Massachusetts	867
California	2,455	Ohio	1,345	Florida	740
Illinois	2,157	New Jersey	1,152	Wisconsin	734
Pennsylvania	1,896	Indiana	916	Canada	670
Michigan	1,852	Minnesota	889	Missouri	549

LEADERS IN MEMBERSHIP TOTALS
(Not Including Subscriptions)

New York	2,128	Ohio	1,091	Massachusetts	640
Illinois	1,622	Texas	1,040	Florida	547
California	1,493	New Jersey	835	Wisconsin	547
Pennsylvania	1,437	Indiana	753	Kansas	425
Michigan	1,247	Minnesota	690	Missouri	423

LEADERS IN MEMBERSHIP GROWTH
(Including Subscriptions)

New York	709	Indiana	293	Massachusetts	219
California	690	Georgia	279	New Jersey	191
Michigan	652	Minnesota	267	North Carolina	185
Pennsylvania	410	Ohio	246	Texas	165
Illinois	347	Canada	219	Maryland	139

STATES AND TERRITORIES HAVING REACHED THEIR GOALS OR GONE BEYOND THEM
(Based on the 25,000 Goal)

Arizona	339%	Colorado	134%	Kentucky	117%
Nevada	200%	Washington	134%	Louisiana	113%
Canada	200%	Minnesota	133%	New Mexico	113%
Oregon	197%	Connecticut	132%	Alabama	112%
Michigan	185%	Idaho	130%	Maryland	111%
Hawaii	180%	Texas	127%	Illinois	109%
Utah	180%	Florida	126%	Ohio	109%
North Dakota	176%	Pennsylvania	124%	North Carolina	108%
Georgia	170%	New Jersey	123%	Iowa	106%
California	167%	Rhode Island	122%	Vermont	106%
South Dakota	150%	New Hampshire	120%	Wisconsin	105%
Montana	148%	Oklahoma	120%	Indiana	102%
New York	147%	Maine	119%	Mississippi	102%
Wyoming	138%	Massachusetts	118%		

STATES AND TERRITORIES HAVING REACHED THEIR GOALS OR GONE BEYOND THEM
(Based on 40,000 Goal)

Arizona	212%	Michigan	116%	North Dakota	110%
Nevada	126%	Hawaii	114%	Georgia	106%
Canada	125%	Utah	113%	California	105%
Oregon	123%				

LEADERS IN PER CENT OF GROWTH
(Compared with May 1959 Membership)

Georgia	Utah	Minnesota	West Virginia
Hawaii	Canada	Nebraska	Colorado
North Dakota	Alabama	Oregon	Washington
North Carolina	Indiana	Rhode Island	Maine
Michigan	New Mexico	California	Maryland

STATES AND TERRITORIES WITH CONTINUOUS GROWTH
(Since May 1959)

Alabama	Maryland	New Jersey	Utah
Arizona	Massachusetts	New York	Vermont
California	Minnesota	North Carolina	Washington
Georgia	Missouri	Ohio	West Virginia
Hawaii	Montana	South Carolina	Wisconsin
Iowa	Nebraska	Tennessee	Wyoming

STATES AND TERRITORIES WITH THE GREATEST NUMBER OF JUNIOR MEMBERS

Michigan	414	Indiana	128	Canada	54
Pennsylvania	301	New Jersey	127	Colorado	47
New York	273	Ohio	124	Hawaii	46
Illinois	251	Texas	108	Louisiana	46
California	149	Wisconsin	101	Massachusetts	44
Minnesota	139	Oregon	71	Missouri	44
Arizona	130	Iowa	64		

Analysis of Membership Growth, Members and Subscribers, May 1959 to May 1960

	May 1959	May 1960		Goal and Per Cent of Goal Reached	
		Individuals	Totals	25,000 Goal	Per Cent
Alabama	214	221	316	282	112%
Alaska	17	12	19	20	95%
Arizona	278	265	339	100	339%
Arkansas	197	155	227	285	80%
California	1,765	1,493	2,455	1,465	167%
Colorado	294	319	402	300	134%
Connecticut	386	357	503	380	132%
Delaware	87	83	111	117	95%
District of Columbia	237	208	239	307	78%
Florida	624	547	740	585	126%
Georgia	248	272	527	310	170%
Hawaii	65	99	137	76	180%
Idaho	38	28	39	30	130%
Illinois	1,810	1,622	2,157	1,980	109%
Indiana	623	753	916	899	102%
Iowa	391	390	522	492	106%
Kansas	411	425	513	517	99%
Kentucky	188	185	240	205	117%
Louisiana	398	351	518	460	113%
Maine	101	111	137	115	119%
Maryland	390	383	529	475	111%
Massachusetts	648	640	867	732	118%
Michigan	1,200	1,248	1,852	1,000	185%
Minnesota	622	690	889	667	133%
Mississippi	175	159	209	205	102%
Missouri	438	423	549	550	99%
Montana	117	120	148	100	148%
Nebraska	188	208	268	275	97%
Nevada	42	29	44	22	200%
New Hampshire	106	102	128	107	120%
New Jersey	961	835	1,152	935	123%
New Mexico	114	112	166	147	113%
New York	2,184	2,128	2,893	1,967	147%
North Carolina	274	285	459	425	108%
North Dakota	68	101	127	72	176%
Ohio	1,099	1,091	1,345	1,232	109%
Oklahoma	375	327	472	395	120%
Oregon	306	298	437	222	197%
Pennsylvania	1,486	1,437	1,896	1,532	124%
Rhode Island	101	109	144	117	122%
South Carolina	160	144	207	227	91%
South Dakota	79	61	90	60	150%
Tennessee	281	258	339	410	83%
Texas	1,260	1,040	1,425	1,120	127%
Utah	96	95	144	80	180%
Vermont	66	55	71	67	106%
Virginia	456	408	519	595	87%
Washington	374	270	511	380	134%
West Virginia	133	152	183	262	70%
Wisconsin	626	547	734	700	105%
Wyoming	63	55	83	60	138%
TOTALS	22,795	21,706	29,937	24,063	124%
U. S. Possessions	33	25	36	40	90%
Canada	451	385	670	335	200%
Foreign	446	249	551	565	98%
GRAND TOTALS	23,790	22,365	31,194	25,003	125%

Analysis of Membership Growth, May 1959 to May 1960

	May 1959	May 1960	Increase	Per Cent Increase	40,000 Goal	Comparison with Goal
Alabama	214	316	102	48%	451	−135
Alaska	17	19	2	12%	32	−13
Arizona	278	339	61	22%	160	+179
Arkansas	197	227	30	16%	456	−229
California	1,765	2,455	690	39%	2,344	+111
Colorado	294	402	108	37%	480	−78
Connecticut	386	503	117	30%	608	−105
Delaware	87	111	24	28%	187	−76
District of Columbia	237	239	2	1%	491	−252
Florida	624	740	116	19%	936	−196
Georgia	248	527	279	113%	496	+31
Hawaii	65	137	72	111%	121	+16
Idaho	38	39	1	3%	48	−9
Illinois	1,810	2,157	347	19%	3,168	−1,011
Indiana	623	916	293	47%	1,443	−527
Iowa	391	522	131	34%	787	−265
Kansas	411	513	102	25%	827	−214
Kentucky	188	240	52	28%	328	−88
Louisiana	398	518	120	30%	736	−218
Maine	101	137	36	36%	184	−47
Maryland	390	529	139	36%	760	−231
Massachusetts	648	867	219	34%	1,171	−304
Michigan	1,200	1,852	652	54%	1,000	+852
Minnesota	622	889	267	43%	1,068	−179
Mississippi	175	209	34	19%	328	−119
Missouri	438	549	111	25%	880	−331
Montana	117	148	31	27%	160	−12
Nebraska	188	268	80	43%	440	−172
Nevada	42	44	2	5%	35	+9
New Hampshire	106	128	22	21%	171	−43
New Jersey	961	1,152	191	20%	1,496	−344
New Mexico	114	166	52	46%	235	−69
New York	2,184	2,893	709	32%	3,147	−254
North Carolina	274	459	185	68%	680	−221
North Dakota	68	127	59	87%	115	+12
Ohio	1,099	1,345	246	22%	1,971	−626
Oklahoma	375	472	97	26%	632	−160
Oregon	306	437	131	43%	355	+82
Pennsylvania	1,486	1,896	410	28%	2,451	−555
Rhode Island	101	144	43	43%	187	−43
South Carolina	160	207	47	29%	363	−156
South Dakota	79	90	11	14%	96	−6
Tennessee	281	339	58	21%	656	−317
Texas	1,260	1,425	165	13%	1,792	−367
Utah	96	144	48	50%	128	+16
Vermont	66	71	5	8%	107	−36
Virginia	456	519	63	14%	952	−433
Washington	374	511	137	37%	608	−97
West Virginia	133	183	50	38%	419	−236
Wisconsin	626	734	108	17%	1,120	−386
Wyoming	63	83	20	32%	96	−13
TOTALS	22,795	29,937	7,142	31%	38,502	−8,565
U. S. Possessions	33	36	3	9%	64	−28
Canada	451	670	219	49%	536	+134
Foreign	446	551	105	24%	904	−353
GRAND TOTALS	23,790	31,194	7,404	31%	40,006	−8,812

THE ARITHMETIC TEACHER

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(INCORPORATED)

Classified Index
Volume VII
1960

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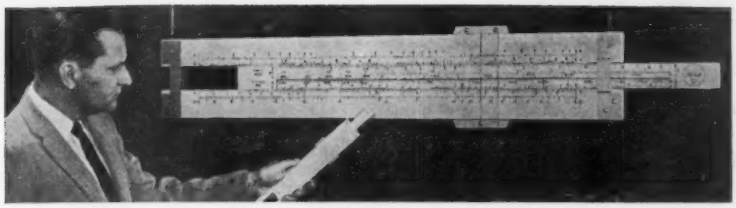

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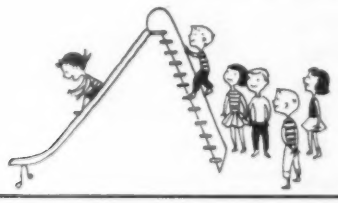

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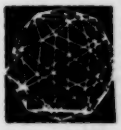

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